**Bookkeeper Rule**

**Multinomial Theorem**

# permutations of the word `bookkeeper`?

- # perms bo o k k e e p e r = 10!
- map perm o b e o k r k e e p e to obeokrkepe
- 2 o’s, 2 k’s, 3 e’s: map is 2!·2!·3!-to-1

\[
\binom{n}{n_1, n_2, \ldots, n_k} := \frac{n!}{n_1! n_2! \cdots n_k!}
\]

**Multinomial coefficient**

Binomial coefficients

**Binomial a special case:**

\[
\binom{n}{k} = \binom{n}{k, n-k}
\]
multinomials

What is the coefficient of \( \text{EMS}^3\text{TY} \) in the expansion of \((E + M + S + T + Y)^7\)?

The number of ways to rearrange the letters in the word SYSTEMS.

multinomial coefficients

What is the coefficient of \( \text{BA}^3\text{N}^2 \) in the expansion of \((B + A + N)^6\)?

The number of ways to rearrange the letters in the word BANANA.

applying the BOOKKEEPER rule

What is the coefficient of \( \text{EMS}^3\text{TY} \) in the expansion of \((E + M + S + T + Y)^7\)?

\[
\begin{pmatrix} 7 \\ 1,1,3,1,1 \end{pmatrix}
\]

multinomial coefficients

What is the coefficient of \( \text{BA}^3\text{N}^2 \) in the expansion of \((B + A + N)^6\)?

\[
\begin{pmatrix} 6 \\ 1,3,2 \end{pmatrix}
\]
multinomial coefficients

Take a 14-mile walk including 3 Northward miles, 4 Southward, 5 Eastward and 3 Westward. How many different walks?

\[ = \#\text{rearrangements of } N^3 S^4 E^5 W^2 \]

\[ = \binom{14}{3, 4, 5, 2} \]

The Multinomial Formula

\[ (X_1 + X_2 + \ldots + X_k)^n = \sum_{r_1 + \ldots + r_k = n} \binom{n}{r_1, r_2, r_3, \ldots, r_k} X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k} \]

multinomial coefficients

What is the coefficient of \( X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k} \) in the expansion of \( (X_1 + X_2 + X_3 + \ldots + X_k)^n \)?

\[ \binom{n}{r_1, r_2, r_3, \ldots, r_k} \]

:= 0 if \( r_1 + r_2 + \ldots + r_k \neq n \)