

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

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# Asymptotic Notation



Albert R Meyer,

April 10, 2013

theOhs.1

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## Asymptotic Equivalence

Def:  $f(n) \sim g(n)$

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$$



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## Asymptotic Equivalence $\sim$

$$n^2 \sim n^2 + n$$

because

$$\lim_{n \rightarrow \infty} \frac{n^2 + n}{n^2} = \lim_{n \rightarrow \infty} 1 + \frac{1}{n} = 1$$



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## Asymptotic Equivalence $\sim$

Lemma:  $\sim$  is symmetric

Proof: Say  $f \sim g$ . Now

$$\lim \frac{g}{f} = \lim \frac{1}{\left(\frac{f}{g}\right)} = \frac{1}{\lim \left(\frac{f}{g}\right)} = \frac{1}{1}$$



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## Asymptotic Equivalence $\sim$

Lemma:  $\sim$  is symmetric

Proof: so  $g \sim f$ . ■

$$\lim \frac{g}{f} = \lim \frac{1}{\left(\frac{f}{g}\right)} = \frac{1}{\lim \left(\frac{f}{g}\right)} = \frac{1}{1}$$



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## transitivity of $\sim$

Suppose  $f \sim g$  and  $g \sim h$ ,  
prove  $f \sim h$ .

$$1 = \lim \frac{f}{g} = \lim \frac{\left(\frac{f}{h}\right)}{\left(\frac{g}{h}\right)} = \frac{\lim \left(\frac{f}{h}\right)}{\lim \left(\frac{g}{h}\right)}$$



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## transitivity of $\sim$

Suppose  $f \sim g$  and  $g \sim h$ ,  
prove  $f \sim h$ .

$$1 = \lim \frac{f}{g} = \lim \frac{\left(\frac{f}{h}\right)}{\left(\frac{g}{h}\right)} = \frac{\lim \left(\frac{f}{h}\right)}{1}$$



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## Asymptotic Equivalence $\sim$

Corollary:  $\sim$  is an  
equivalence relation



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Asymptotic Equivalence  $\sim$

$\sim$  is a relation  
on functions:

$$f \sim g$$



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Little Oh:  $o(\cdot)$

Asymptotically smaller

Def:  $f(n) = o(g(n))$

iff

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$$



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Little Oh:  $o(\cdot)$

$$n^2 = o(n^3)$$

because

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^3} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$



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6	9	13	7
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Little Oh:  $o(\cdot)$

Lemma:

$o(\cdot)$  is a strict  
partial order



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Big Oh:  $O(\cdot)$

Asymptotic Order of Growth:

$$f = O(g)$$

$$\limsup_{n \rightarrow \infty} \left( \frac{f(n)}{g(n)} \right) < \infty$$

a technicality — ignore now



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Big Oh:  $O(\cdot)$

$$3n^2 = O(n^2)$$

because

$$\lim_{n \rightarrow \infty} \frac{3n^2}{n^2} = 3 < \infty$$



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Theta:  $\Theta(\cdot)$

Same Order of Growth:

$$f = \Theta(g)$$

$$\text{Def: } f = O(g)$$

and

$$g = O(f)$$



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Theta:  $\Theta(\cdot)$

Lemma:

$\Theta(\cdot)$  is an  
equivalence  
relation



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## Asymptotics: Intuitive Summary

$f \sim g$ :  $f$  &  $g$  nearly equal

$f = o(g)$ :  $f$  much less than  $g$

$f = O(g)$ :  $f$  roughly  $\leq g$

$f = \Theta(g)$ :  $f$  roughly equal  $g$

