Mathematics for Computer Science
MIT 6.042J/18.062J

Asymptotic Notation

Asymptotic Equivalence

Def: $f(n) \sim g(n)$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

Asymptotic Equivalence ~

$$n^2 \sim n^2 + n$$

because

$$\lim_{n\to\infty} \frac{n^2+n}{n^2} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1$$

Asymptotic Equivalence ~

Lemma: ~ is symmetric

Proof: Say $f \sim g$. Now

$$\lim \frac{g}{f} = \lim \frac{1}{\left(\frac{f}{g}\right)} = \frac{1}{\lim\left(\frac{f}{g}\right)} = \frac{1}{1}$$

Asymptotic Equivalence ~

Lemma: ~ is symmetric

Proof: so $g \sim f$. ■

$$\lim \frac{g}{f} = \lim \frac{1}{\left(\frac{f}{g}\right)} = \frac{1}{\lim\left(\frac{f}{g}\right)} = \frac{1}{1}$$

transitivity of ~

Suppose $f \sim g$ and $g \sim h$,

prove $f \sim h$.

$$1 = \lim \frac{f}{g} = \lim \frac{\left(\frac{f}{h}\right)}{\left(\frac{g}{h}\right)} = \frac{\lim\left(\frac{f}{h}\right)}{\lim\left(\frac{g}{h}\right)}$$

transitivity of ~

Suppose $f \sim g$ and $g \sim h$,

prove $f \sim h$.

$$1 = \lim \frac{f}{g} = \lim \frac{\left(\frac{f}{h}\right)}{\left(\frac{g}{h}\right)} = \frac{\lim\left(\frac{f}{h}\right)}{1}$$

Asymptotic Equivalence ~

Corollary: ~ is an equivalence relation

Asymptotic Equivalence ~

~ is a relation on **functions**:

$$f \sim g$$

Little Oh: $o(\cdot)$

Asymptotically smaller

Def: $f(n) = o(g(n))$

iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

Little Oh: $o(\cdot)$

$$n^2 = o(n^3)$$

because

$$\lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Little Oh: $o(\cdot)$

Lemma:

$o(\cdot)$ is a strict partial order

Big Oh: $O(\cdot)$

Asymptotic Order of Growth:

$$f = O(g)$$

$$\lim_{n\to\infty} \cancel{\sup} \left(\frac{f(n)}{g(n)}\right) < \infty$$

a technicality —ignore now

Big Oh: $O(\cdot)$

$$3n^2 = O(n^2)$$

because

$$\lim_{n\to\infty} \frac{3n^2}{n^2} = 3 < \infty$$

Theta: $\Theta(\cdot)$

Same Order of Growth:

$$f = \Theta(g)$$

Def: $f = O(g)$

and

$g = O(f)$

Theta: $\Theta(\cdot)$

Lemma:

$\Theta(\cdot)$ is an equivalence relation

Asymptotics: Intuitive Summary

$f \sim g$: f & g nearly equal

$f = o(g)$: f much less than g

$f = O(g)$: f roughly $\leq g$

$f = \Theta(g)$: f roughly equal g