Mathematics for Computer Science
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3-Coloring Reduces to SAT

A Simple Graph $G$

Propositional Variables
Want a formula that means $G$ is 3-colorable
Do this using variables $R_1$, $W_1$, $B_1$
which will mean that vertex 1 is colored Red, White, or Blue

Propositional Formula

$[R_1 \text{ OR } W_1 \text{ OR } B_1]$

AND

which will mean that vertex 1 is colored Red, White, or Blue
Propositional Formula

\[
[(R_1 \text{ AND } W_1 \text{ AND } B_1) \text{ OR } (W_1 \text{ AND } R_1 \text{ AND } B_1) \text{ OR } (B_1 \text{ AND } W_1 \text{ AND } R_1)]
\]

and vertex 1 has only one color

Propositional Formula

AND vertex 1 has a different color than vertex 6

\[
\text{NOT}(R_1 \text{ AND } R_6) \text{ AND } \text{NOT}(W_1 \text{ AND } W_6) \text{ AND } \text{NOT}(B_1 \text{ AND } B_6)
\]

Propositional Formula

Do the same for vertices 2-6 and for the remaining six edges. Let

\[\text{Prop}_G\]

be the AND of all these formulas

SAT vs 3-Color

\[G\] is 3-colorable so setting

\[
R_1 R_4 W_3 W_6 B_2 B_5
\]

True

and the rest False satisfies \[\text{Prop}_G\]
For any simple graph $G$ construct $\text{Prop}_G$ similarly. $G$ is 3-colorable iff $\text{Prop}_G$ is satisfiable.

3-Color reduces to SAT: a good SAT procedure would yield a good 3-Coloring procedure (and conversely).

SAT and 3-Color stand and fall together: there is an "efficient" (polynomial time) SAT procedure iff there is one for 3-Color. Both problems are NP-complete.