Staff Solutions to Midterm Exam February 21

Problem 1 (Irrational logarithm) (20 points).
Prove that \( \log_5 10 \) is irrational.

**COMMENTS:**
- variant of MQ_log12_of_18_irrational

**keywords** = [ irrational power contradiction log ]

**Solution.** Proof. Suppose to the contrary that

\[
\log_5 10 = \frac{m}{n}
\]

for some integers \( m, n \) where \( n > 0 \). So we have

\[
\begin{align*}
5^{\log_5 10} &= \frac{m}{n} \quad \text{(raising 5 to equal powers),} \\
10 &= \frac{m}{n} \quad \text{(def of } \log_5 ) \,, \\
10^n &= \frac{m^n}{n} \quad \text{(raising both sides to the } n^{th} \text{ power),} \\
(2 \cdot 5)^n &= s^n \quad \text{(factoring 10 into primes),} \\
2^n \cdot 5^n &= s^m.
\end{align*}
\]

By the uniqueness of prime factorization, the powers of primes on the left and right hand sides of the last equation must be equal. Since there is no factor of 2 on the right hand side, \( n \) must equal 0, contradicting the definition of \( n \). Hence \( \log_5 10 \) cannot be rational.

Problem 2 (Impossible Equality by Well ordering) (25 points).
Use the Well Ordering Principle to prove that there is no solution over the positive integers to the equation:

\[
4a^3 + 2b^3 = c^3.
\]

**COMMENTS:**
- CP_like_Lehmans_equation
- from F05.cp3f
- edited by ARM 3/14/12

**keywords** = [ well_ordering WOP contradiction ]
Solution. Let $S$ be the set of all positive integers $a$ for which there are positive integers $b$ and $c$ that satisfy this equation. We want to show that $S$ is empty.

Assume to the contrary that $S$ is nonempty. Then by the well-ordering principle, $S$ contains a smallest element $a_0$. By the definition of $S$, there are corresponding positive integers $b_0$ and $c_0$ such that:

$$4a_0^3 + 2b_0^3 = c_0^3.$$ 

The left side of this equation is even, so $c_0^3$ is even, and therefore $c_0$ is also even. Thus, there exists an integer $c_1$ such that $c_0 = 2c_1$. Substituting $2c_1$ for $c_0$ in the preceding equation and then dividing both sides by 2 gives:

$$2a_0^3 + b_0^3 = 4c_1^3.$$ 

Now $b_0^3$ must be even, so $b_0$ is even. Thus, there is an integer $b_1$ such that $b_0 = 2b_1$. Substituting for $b_0$ in the preceding equation and dividing both sides by 2 gives:

$$a_0^3 + 4b_1^3 = 2c_1^3.$$ 

From this equation, we know that $a_0^3$ is even, so $a_0$ is also even. Thus, there exists an integer $a_1$ such that $a_0 = 2a_1$. Finally, substituting for $a_0$ in the previous equation and dividing by 2 gives

$$4a_1^3 + 2b_1^3 = c_1^3.$$ 

So $a = a_1$, $b = b_1$ and $c = c_1$ is another solution to the original equation, which means that $a_1 \in S$ by definition. But $a_1 < a_0$, contradicting the fact that $a_0$ is the smallest element of $S$.

This contradiction implies that $S$ must be empty, which means the original equation has no solutions over the positive integers. 

Problem 3 (Tetris Formula by Well Ordering) (30 points).

Mini-Tetris is a game whose objective is to provide a complete “tiling” of a $2 \times n$ board using tiles of specified shapes. In this problem we consider the following set of five tiles:

For example, there are two possible tilings of a $2 \times 1$ board:

Also, here are three tilings for a $2 \times 2$ board:
Note that tiles may not be rotated, which is why the second and third of the above tilings count as different, even though one is a 180° rotation of the other. (A 90° degree rotation of these shapes would not count as a tiling at all.)

(a) There are four more 2 × 2 tilings in addition to the three above. What are they?

(b) $T_n$ can be specified in terms of $T_{n-1}$ and $T_{n-2}$ as follows:

$$T_n = 2T_{n-1} + 3T_{n-2}$$

for $n \geq 2$.

Briefly explain how to justify this equation.

Solution. Every winning tiling on a $2 \times n$ board is of one of the following five types:
There are $T_{n-1}$ tilings for each of the first two types and $T_{n-2}$ tilings for each of the last three types, so the total number of $2 \times n$ tilings is given by the right hand side of equation 2.

(c) Use the Well Ordering Principle to prove that for $n \geq 0$, the number $T_n$ of tilings of a $2 \times n$ Mini-Tetris board is:

$$\frac{3^{n+1} + (-1)^n}{4}.$$  

Solution. Let $P(n)$ be the predicate

$$P(n) ::= \left[ T_n = \frac{3^{n+1} + (-1)^n}{4} \right].$$

and let $C$ be the set of counterexamples to $P$:

$$C ::= \{ n \geq 0 \mid \text{NOT}(P(n)) \}.$$  

Assume for the sake of contradiction that $C$ is not empty. Then by the Well Ordering Principle, there is some minimum element $m \in C$. But $P(n)$ is true for $n = 0, 1$:

$$T_0 = 1 = \frac{3^{0+1} + (-1)^0}{4},$$

$$T_1 = 2 = \frac{3^{1+1} + (-1)^1}{4}.$$  

This means that $m$ must be greater than 1. So both $m - 1$ and $m - 2$ are $\geq 0$, and since $m$ is the smallest nonnegative counterexample, neither $m - 1$ nor $m - 2$ is a counterexample. That is, $P(m - 1)$ and $P(m - 2)$ are true.

Thus, we have

$$T_m = 2T_{m-1} + 3T_{m-2},$$

(by 2)

$$= 2 \frac{3^m + (-1)^{m-1}}{4} + 3 \frac{3^{m-1} + (-1)^{m-2}}{4}$$

(by $P(m - 1)$ and $P(m - 2)$)

$$= \frac{2(3^m + (-1)^{m-1}) + 3(3^{m-1} + (-1)^{m-2})}{4}$$

$$= \frac{2(3^m) + 3^m + (-2 + 3)(-1)^m}{4}$$

$$= \frac{3^{m+1} + (-1)^m}{4}.$$  

This shows that $P(m)$ is true, contradicting the definition of $m$. So $C$ must be empty, which proves that $P(n)$ is true for all $n \geq 0$, as desired.

Problem 4 (Validity by Cases) (25 points).

The formula

$$\text{NOT}(\overline{A} \text{ IMPLIES } B) \text{ AND } A \text{ AND } C$$

$$\text{IMPLIES}$$

$$D \text{ AND } E \text{ AND } F \text{ AND } G \text{ AND } H \text{ AND } I \text{ AND } J \text{ AND } K \text{ AND } L \text{ AND } M$$

turns out to be valid.
(a) Explain why verifying the validity of this formula by truth table would be very hard for one person to do with pencil and paper (no computers).

**COMMENTS:**
- FP_validity_by_cases
- ARM 2/17/17

**keywords** = [ *validity cases truth_table* ]

**Solution.** The number of entries in a truth table here would be \(2^{14}\) or about 16,000 since there are 14 variables. This could take weeks for one person to do by hand.

(b) Verify that the formula is valid, reasoning by cases according to the truth value of \(A\).

**Proof.** **Case:** (\(A\) is True).

**Solution.** Since the implication **True** IMPLIES **B** is **True**, the expression NOT(\(\overline{A}\) IMPLIES **B**) evaluates to **False**. So the hypothesis (upper formula) of the main implication is **False**, making the whole implication formula **True**. So the formula is **True** in all assignments with \(A\) assigned **True**.

**Case:** (\(A\) is False).

**Solution.** The hypothesis (upper formula) side of the formula is now equivalent to (**False** AND \(\ldots\)), and so is immediately **False**. As in the previous case, this makes the whole implication formula **True**. So the formula is **True** in all assignments with \(A\) assigned **False**.

Since \(A\) must be **True** or **False** in any truth assignment, the formula is **True** in any case. That is, it is valid.