Staff Solutions to In-Class Problems Week 8, Fri.

STAFF NOTE: Simple Graphs: Degree & Isomorphism

Half the teams finished in 45 min. Asked them to work out the 4 possible isomorphisms for 2(b), and gave a sermon about “buildup error” re problem 3. Then asked for comments on midterm 3. Most still dismissed after one hour.

Problem 1.
Which of the items below are simple-graph properties preserved under isomorphism?
(a) There is a cycle that includes all the vertices.

STAFF NOTE: If asked, explain that simple graph cycles can be defined in the essentially same way as for digraphs. The only difference is that going back and forth on the same edge—a length 2 “cycle”—is not considered to be a cycle.

COMMENTS:
- TP_preserved_under_isomorphism
- overlaps FP_graphs_short_answer, FP_multiple_choice_unhidden
- by ARM 3/29/13 for first simple graph lecture
- solns updates ARM 10/31/15

keywords = [ simple_graph isomorphism vertices preserved degree edge path ]

Solution. Preserved.

(b) The vertices are numbered 1 through 7.

Solution. Not a property of simple graphs.
Numbers or labels may be assigned to vertices for various purposes, but vertex numbering is something added to a simple graph but is not a “built in” property of simple graphs.

(c) The vertices can be numbered 1 through 7.

Solution. Preserved.
The vertices can be numbered 1 through 7 iff the graph has 7 vertices, which is a property that is preserved under isomorphism.

(d) There are two degree 8 vertices.

Solution. Preserved.
(e) Two edges are of equal length.

**Solution.** “Length” is not a property of edges in a simple graph. (Edges in a *drawing* of the graph may or may not have the same length, depending on how they are drawn.)

(f) No matter which edge is removed, there is a path between any two vertices.

**Solution.** Preserved.

(g) There are two cycles that do not share any vertices.

**Solution.** Preserved.

(h) The vertices are sets.

**Solution.** NOT Preserved.

Isomorphism does not preserve what things are made of, only how they fit together.

However, this question was intended to remind you of the Theorem that every DAG is isomorphic to a DAG whose vertices are sets.

(i) The graph can be drawn in a way that all the edges have the same length.

**Solution.** Preserved.

If you can draw a simple graph in a certain way, then you can draw any isomorphic graph in the same way.

(j) No two edges cross.

**Solution.** “Crossing” is not a property of edges in a simple graph.

On the bother hand, edges in a *drawing* of a graph may or may not cross, depending on how they are drawn. The property that a graph *can be drawn* in the plane without edges crossing is preserved under isomorphism. It is an important enough property that a whole chapter is devoted entirely to such *planar* graphs.

(k) The OR of two properties that are preserved under isomorphism.

**Solution.** Preserved.

**STAFF NOTE:** When problem is done, have students come back a **write out a proof of this case:**

**Proof.** Suppose \( P \) and \( Q \) are graph properties preserved under isomorphism, and \( G \) and \( H \) are isomorphic simple graphs. Then the truth values of both \( P \) and \( Q \) will be the same in \( G \) and \( H \), and so any propositional combination of \( P \) and \( Q \) will also have the same truth value in \( G \) and \( H \).

Second, more pedantic proof:
Proof. Let $R := P$ OR $Q$. Then

$$
\begin{align*}
R(G) & \text{ IFF } P(G) \text{ OR } Q(G) \text{ (def of } R) \\
& \text{ IFF } P(H) \text{ OR } Q(H) \text{ (since } P, Q \text{ are preserved)} \\
& \text{ IFF } R(H) \text{ (def of } R).
\end{align*}
$$

so $R$ is preserved, as claimed. ■

(l) The negation of a property that is preserved under isomorphism.

Solution. Preserved, by the same reasoning as for the OR of two properties. ■

Problem 2.
For each of the following pairs of simple graphs, either define an isomorphism between them, or prove that there is none. (We write $ab$ as shorthand for $(a \rightarrow b)$.)

(a)

$G_1$ with $V_1 = \{1, 2, 3, 4, 5, 6\}$, $E_1 = \{12, 23, 34, 14, 15, 35, 45\}$

$G_2$ with $V_2 = \{1, 2, 3, 4, 5, 6\}$, $E_2 = \{12, 23, 34, 45, 51, 24, 25\}$

COMMENTS:

- CP_isomorphic_graphs
- from: S09.cp6m, S06.cp5w

keywords = [ isomorphism isomorphic digraph ]

Solution. Not isomorphic: $G_2$ has a node, 2, of degree 4, but the maximum degree in $G_1$ is 3.

So if as in this example, two graphs have different degree “spectra,” then they are clearly not isomorphic. On the other hand, there are simple examples of non-isomorphic graphs with the same degree spectra.

STAFF NOTE: Ask students to come up with an example of their own before showing them the following:

For example,

- a (disconnected) graph consisting of two triangles, and
- a hexagon,

are non-isomorphic graphs, each with six vertices all of which have degree two.

Even without finding such a simple counterexample, you could have concluded that it would be too good to be true for degree spectra to determine isomorphism. If having the same degree spectra implied isomorphism, the isomorphism would be easy to check. But it is a long standing open problem to find an easy way to verify isomorphism.

(b)

$G_3$ with $V_3 = \{1, 2, 3, 4, 5, 6\}$, $E_3 = \{12, 23, 34, 14, 45, 56, 26\}$

$G_4$ with $V_4 = \{a, b, c, d, e, f\}$, $E_4 = \{ab, bc, cd, de, ae, ef, cf\}$
Solution. STAFF NOTE: Supppemental question: have students describe all the isomorphisms. 

Isomorphic (four isomorphisms) with the vertex correspondences:
1f, 2c, 3d, 4e, 5a, 6b or
1f, 2e, 3d, 4c, 5b, 6a or
1d, 2c, 3f, 4e, 5a, 6b or
1d, 2e, 3f, 4c, 5b, 6a.

Problem 3.
Let’s say that a graph has “two ends” if it has exactly two vertices of degree 1 and all its other vertices have degree 2. For example, here is one such graph:

(a) A line graph is a graph whose vertices can be listed in a sequence with edges between consecutive vertices only. So the two-ended graph above is also a line graph of length 4.

Prove that the following theorem is false by drawing a counterexample.
False Theorem. Every two-ended graph is a line graph.

COMMENTS:

- PS_bogus_graph_two_ends
- was called PS_graph_two_ends
- spring07 pset4-8, S15.ps7
- slightly revised to not mention paths, by ARM 10/8/09

keywords = [ graph theory induction buildup_error bogus_proof ]

Solution. A graph consisting of a path together with a cycle is a counterexample.

(b) Point out the first erroneous statement in the following bogus proof of the false theorem and describe the error.

Bogus proof. We use induction. The induction hypothesis is that every two-ended graph with $n$ edges is a line graph.

Base case ($n = 1$): The only two-ended graph with a single edge consists of two vertices joined by an edge:

Sure enough, this is a line graph.

Inductive case: We assume that the induction hypothesis holds for some $n \geq 1$ and prove that it holds for $n + 1$. Let $G_n$ be any two-ended graph with $n$ edges. By the induction assumption, $G_n$ is a line graph. Now suppose that we create a two-ended graph $G_{n+1}$ by adding one more edge to $G_n$. This can be done in only one way: the new edge must join one of the two endpoints of $G_n$ to a new vertex; otherwise, $G_{n+1}$ would not be two-ended.

Clearly, $G_{n+1}$ is also a line graph. Therefore, the induction hypothesis holds for all graphs with $n + 1$ edges, which completes the proof by induction.
Solution. STAFF NOTE: The problem is to identify the mistake in the proof above, not to explain how to do a correct proof of some related true assertion. Make sure students attend to locating the mistake. ■

Actually, this is a correct proof of something else. That is, the first erroneous statement is the last one claiming that the induction hypothesis holds for all \((n + 1)\)-edge two-ended graphs.

The proof doesn’t show this; rather, it only shows that the induction hypothesis holds for those two-ended \((n + 1)\)-edge graphs that can be obtained by adding one more edge to an \(n\)-edge line graph. But not all two-ended graphs can be built in this way, as the counterexample demonstrates.

STAFF NOTE: Get a discussion going about buildup error: ■

This is an example of “buildup” error, where you assume that a size \(n + 1\) object is built up in some particular way from similar objects of smaller size. (This assumption is correct for some kinds of objects, but incorrect for others—such as the one in the argument above.)

One way to avoid an accidental buildup error is to use a “shrink down, grow back” process in the inductive step: start with a size \(n + 1\) object, say a graph, remove a vertex (or edge), apply the inductive hypothesis \(P(n)\) to the smaller graph, and then add back the vertex (or edge) and argue that \(P(n + 1)\) holds. Let’s see what would have happened if we’d tried to prove the claim above by this method:

**Inductive step:** We must show that \(P(n)\) implies \(P(n + 1)\) for all \(n \geq 1\). Consider an \((n + 1)\)-vertex two-ended graph \(G\). Remove an end vertex, to be left with \(n\)-vertex graph \(G'\) which should still be two-ended … uh-oh it might not be! The reduced graph \(G'\) might contain a vertex of degree 0, making the inductive hypothesis \(P(n)\) inapplicable! We are stuck—and properly so, since the claim is false! ■

Problem 4.
The average degree of the vertices in an \(n\)-vertex graph is twice the average number of edges per vertex. Explain why.

**COMMENTS:**

- TP_handshake_average
- renamed from MQ_
- ARM 10/21/13

**keywords** = [handshaking degree simple_graph average]

**Solution.** Let \(a_E\) be the average number of edges per vertex. So by definition

\[a_E := \frac{|E|}{n}.\]
The average degree $d_V$ of a vertex is by definition the sum of the vertex degrees divided by $n$. That is,

$$d_V := \frac{\sum_{v \in V} \deg(v)}{n}$$

$$= \frac{2|E|}{n} \quad \text{(Handshake Lemma)}$$

$$= 2 \frac{|E|}{n} = 2a_E.$$