Staff Solutions to In-Class Problems Week 12, Wed.

STAFF NOTE: Conditional probability, Independence of 2 events Ch.18-18.7

Problem 1.
There is an unpleasant, degenerative disease called Beaver Fever which causes people to tell math jokes unrelentingly in social settings, believing other people will think they’re funny. Fortunately, Beaver Fever is rare, afflicting only about 1 in 1000 people. Doctor Meyer has a fairly reliable diagnostic test to determine who is going to suffer from this disease:

- If a person will suffer from Beaver Fever, the probability that Dr. Meyer diagnoses this is 0.99.
- If a person will not suffer from Beaver Fever, the probability that Dr. Meyer diagnoses this is 0.97.

Let $B$ be the event that a randomly chosen person will suffer Beaver Fever, and $Y$ be the event that Dr. Meyer’s diagnosis is “Yes, this person will suffer from Beaver Fever,” with $\overline{B}$ and $\overline{Y}$ being the complements of these events.

(a) The description above explicitly gives the values of the following quantities. What are their values?

\[
\begin{align*}
\Pr[B] & = 0.001 \\
\Pr[Y \mid B] & = 0.99 \\
\Pr[\overline{Y} \mid B] & = 0.97 \\
\end{align*}
\]

COMMENTS:

- FP\_conditional\_beaver\_fever
- from F07.mq-nov28
- Revisions on grammar, layout, clarity 4/27/14
- ARM added discussion in last part solution 5/3/13

keywords = [ probability conditional probability Bayes ]

Solution. $\Pr[B] = 0.001 \quad \Pr[Y \mid B] = 0.99 \quad \Pr[\overline{Y} \mid B] = 0.97$

(b) Write formulas for $\Pr[\overline{B}]$ and $\Pr[Y \mid \overline{B}]$ solely in terms of the explicitly given quantities in part (a)—literally use their expressions, not their numeric values.

Solution. $\Pr[\overline{B}] = 1 - \Pr[B], \Pr[Y \mid \overline{B}] = 1 - \Pr[\overline{Y} \mid \overline{B}]$.

(c) Write a formula for the probability that Dr. Meyer says a person will suffer from Beaver Fever solely in terms of $\Pr[B]$, $\Pr[\overline{B}]$, $\Pr[Y \mid B]$ and $\Pr[Y \mid \overline{B}]$.
**Solution.** By the Total Probability Law:

$$\Pr[Y] = \Pr[Y \mid B] \Pr[B] + \Pr[Y \mid \overline{B}] \Pr[\overline{B}]$$

The values turn out to be $0.99(1/1000) + 0.03(1 - 1/1000) = 0.03096$. 

(d) Write a formula solely in terms of the expressions given in part (a) for the probability that a person will suffer Beaver Fever given that Doctor Meyer says they will.

**STAFF NOTE:** Have students use a calculator to get the actual value.

Solution.

$$\Pr[B \mid Y] = \frac{\Pr[B \text{ and } Y]}{\Pr[Y]} = \frac{\Pr[Y \mid B] \Pr[B]}{\Pr[Y \mid B] \Pr[B] + \Pr[Y \mid \overline{B}] \Pr[\overline{B}]} = \frac{\Pr[Y \mid B] \Pr[B]}{\Pr[Y \mid B] \Pr[B] + (1 - \Pr[Y \mid \overline{B}]) (1 - \Pr[B])}.$$ 

The values turn out to be

$$\Pr[B \mid Y] = \frac{0.99(1/1000)}{0.03096} = \frac{99}{3096} \approx \frac{1}{32}.$$ 

The low probability of actually suffering Beaver Fever even though the (97% accurate) test says you will is because there are way more people who will not suffer the disease than those who will. Among 1000 people, the number of false positives $(0.99 \times 3\%)$ is more than 30 times the number of true positives $(1 \times 99\%)$. So if the test says you will suffer Beaver Fever, it’s probably a false positive.

Of course Dr. Meyer has a recourse to a 99.9% accurate test that has no false positives: simply telling everyone they won’t get Beaver Fever.

Suppose there was a vaccine to prevent Beaver Fever, but the vaccine was expensive or slightly risky itself. If you were sure you were going to suffer from Beaver Fever, getting vaccinated would be worthwhile, but even if Dr. Meyer diagnosed you as a future sufferer of Beaver Fever, the probability you actually will suffer Beaver Fever remains low (about 1/32 by part (d)).

In this case, you might sensibly decide not to be vaccinated—after all, Beaver Fever is not *that* bad an affliction. So the diagnostic test serves no purpose in your case. You may as well not have bothered to get diagnosed. Even so, the test may be useful:

(e) Suppose Dr. Meyer had enough vaccine to treat 2% of the population. If he randomly chose people to vaccinate, he could expect to vaccinate only 2% of the people who needed it. But by testing everyone and only vaccinating those diagnosed as future sufferers, he can expect to vaccinate a much larger fraction people who were going to suffer from Beaver Fever. Estimate this fraction.

**Solution.** $\approx 2/3$.

The test will diagnose about 3% of the population as future sufferers. This 3% will include 99% of the actual sufferers but mostly include people who will not get Beaver Fever—the false positives. By giving
the vaccine at random to only this 3% that are diagnosed as future sufferers, Dr. Meyer will have enough vaccine for 2/3 of them. So he will be able to vaccinate nearly 2/3 of the people who actually need it. So even though the probability that a diagnosed person will suffer Beaver Fever is small, the increased probability (from 1/1000 to about 1/32) provided by the diagnosis has significant public health value.

Problem 2.
There are three prisoners in a maximum-security prison for fictional villains: the Evil Wizard Voldemort, the Dark Lord Sauron, and Little Bunny Foo-Foo. The parole board has declared that it will release two of the three, chosen uniformly at random, but has not yet released their names. Naturally, Sauron figures that he will be released to his home in Mordor, where the shadows lie, with probability \( \frac{2}{3} \).

A guard offers to tell Sauron the name of one of the other prisoners who will be released (either Voldemort or Foo-Foo). If the guard has a choice of naming either Voldemort or Foo-Foo (because both are to be released), he names one of the two with equal probability.

Sauron knows the guard to be a truthful fellow. However, Sauron declines this offer. He reasons that knowing what the guards says will reduce his chances, so he is better off not knowing. For example, if the guard says, “Little Bunny Foo-Foo will be released”, then his own probability of release will drop to \( \frac{1}{2} \) because he will then know that either he or Voldemort will also be released, and these two events are equally likely.

Dark Lord Sauron has made a typical mistake when reasoning about conditional probability. Using a tree diagram and the four-step method, explain his mistake. What is the probability that Sauron is released given that the guard says Foo-Foo is released?

*Hint:* Define the events \( S, F \) and “\( F \)” as follows:

\[
\begin{align*}
\text{“}F\text{”} &= \text{Guard says Foo-Foo is released} \\
F &= \text{Foo-Foo is released} \\
S &= \text{Sauron is released}
\end{align*}
\]

COMMENTS:
- CP\_conditional\_prob\_says\_so\_bug
- similar to MQ\_voldemort\_returns
- from: F05.ps9 problem 2
- rephrased with hint--ARM 12/2/11

**keywords** = [ conditional\_probability tree\_diagram four\_step\_method ]

**Solution.** Sauron’s mistake can be explained as his confusing the two different events \( F \) and “\( F \)”. His observation that \( \Pr[S \mid F] = \frac{1}{2} \) is correct, but that’s the wrong thing to calculate. He should be calculating \( \Pr[S \mid \text{“}F\text{”}] \).

To clarify the error and work out the proper probability, let’s begin by working out the sample space, noting events of interest, and computing outcome probabilities:
Released

The outcomes in each of these events are noted in the tree diagram.

The tree shows how the event $F$ (Foo-foo will be released) is different from the event "$F$" (the guard says Foo-foo will be released). In particular, the probability that Sauron is released, given that Foo-foo is released, is indeed $1/2$:

$$\Pr[S \mid F] = \frac{\Pr[S \cap F]}{\Pr[F]} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6} + \frac{1}{6}} = \frac{1}{2}$$

But the probability that Sauron is released given that the guard actually says so is still $2/3$:

$$\Pr[S \mid "F"] = \frac{\Pr[S \cap "F"]}{\Pr["F"]} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} = \frac{2}{3}$$

So Sauron’s probability of release is unchanged by the guard’s statement.

Problem 3.
Suppose you repeatedly flip a fair coin until you see the sequence HTT or HHT. What is the probability you see the sequence HTT first?

*Hint*: Try to find the probability that HHT comes before HTT conditioning on whether you first toss an H or a T. The answer is not $1/2$.

**COMMENTS:**
- CP_coin_flip_sequences
- second part of PS_coin_flip_sequences

**keywords** = [ conditional_probability total_probability transitive intransitive infinite_tree ]

**Solution.** We apply the standard tree diagram approach where the $n$th level of the tree corresponds to the results of the $n$th coin flip. This tree is infinite, but we need not be intimidated: the tree has a repeating
structure which allows a simple recursive description. Then the Law of Total Probability applied to the recursive tree structure will lead to some simple equations for the desired probability.

To see how this works, suppose our first toss is \( T \). Since neither of our patterns starts with \( T \), the situation relevant to our patterns is the same as at the start. So if we let \( A \) be the tree diagram describing our coin flipping experiment, then the branch corresponding to flipping \( T \) goes to a copy of \( A \). This is illustrated in Figure 1.

![Figure 1](htt_vs_hht.png)

**Figure 1**  
HTT versus HHT.

If our first flip is \( H \), we need to consider different cases based on the subsequent throws. So let \( B \) be the subtree corresponding the first flipping \( H \), and let \( C \) be the subtree corresponding to the second flip also being \( H \). Now if the third flip is \( T \), then we arrive at the leaf where \( HHT \) has occurred first. On the other hand, if the third flip is \( H \), then the first flip is no longer relevant to the length three patterns we are considering, and we are at the same point in progressing toward our patterns as we were after flipping only two \( H \)'s. That is, the \( H \) branch of subtree \( C \) is another copy of \( C \). This is also illustrated in Figure 1. We finish up by observing that if the first three flips are \( HTT \), then we arrive at the leaf where \( HTT \) has occurred first, and if the first three flips are \( HTH \), then the situation is the same as if we first flipped \( H \). So the subtree at after \( HTH \) is the same as \( B \).

This completes the reasoning which led to the tree diagram illustrated in Figure 1.
Now let $E$ be the event that $\text{HTT}$ appears before $\text{HHT}$. Our task is to calculate $\Pr[E]$. Since the tree $A$ describes the start of the coin flipping, we have

$$\Pr[E] = \Pr[E \mid A].$$

Next, by the Law of Total Probability,

$$\Pr[E \mid A] = \Pr[E \mid A] \cdot \Pr[T] + \Pr[E \mid B] \cdot \Pr[H],$$

which implies

$$\Pr[E \mid A] = \Pr[E \mid B].$$

Again by the Law of Total Probability,

$$\Pr[E \mid B] = \Pr[E \mid BTT] \cdot \Pr[TT] + \Pr[E \mid BTH] \cdot \Pr[TH] + \Pr[E \mid BH] \cdot \Pr[H]$$

$$= 1 \cdot \frac{1}{4} + \Pr[E \mid B] \cdot \frac{1}{4} + \Pr[E \mid C] \cdot \frac{1}{2},$$

$$\Pr[E \mid C] = \Pr[E \mid C T] \cdot \Pr[T] + \Pr[E \mid CH] \cdot \Pr[H]$$

$$= 0 \cdot \frac{1}{2} + \Pr[E \mid C] \cdot \frac{1}{2}.$$ (3)

Now from (4) we immediately get

$$\Pr[E \mid C] = 0.$$

Then from (3),

$$\Pr[E \mid B] = \frac{1}{4} + \Pr[E \mid B] \cdot \frac{1}{4},$$

$$\Pr[E \mid B] = \frac{1}{3},$$

and so by (2)

$$\Pr[E \mid A] = \frac{1}{3}.$$ (4)

That is, $\text{HTT}$ appears before $\text{HHT}$ with probability $1/3$.

These kind of events have an amazing intransitivity property: if you pick any pattern of three flips such as $\text{HTT}$, then I can pick a pattern of three flips such as $\text{HHT}$ whose odds of coming up first are better than even. In particular, even if you instead picked the “better” pattern $\text{HHT}$, there is another pattern I can pick that has a more than even chance of appearing before $\text{HHT}$.

So we leave you with an ethical dilemma: is it OK to allow a naive opponent to first choose whichever pattern of three flips he likes best, then you choose your own preferred pattern, after which you bet real money on the coin flip game?