Staff Solutions to In-Class Problems Week 10, Wed.

STAFF NOTE: Sums & Products, Ch. 14-14.5

Problem 1.
We begin with two large glasses. The first glass contains a pint of water, and the second contains a pint of wine. We pour 1/3 of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour 1/3 of a pint of the mix back into the first glass and repeat this pouring back-and-forth process a total of \(n\) times.

(a) Describe a closed-form formula for the amount of wine in the first glass after \(n\) back-and-forth pourings.

STAFF NOTE: If students get caught up in complicated formulas involving the different quantities of wine and water in each glass, point out that the complete state is determined by the number \(w\) of pints in the first glass.

COMMENTS:

- PS_mixing_water_and_wine
- from: S09.ps8

**keywords** = [ geometric_sum asymptotics recurrences closed_form ]

**Solution.** The state of the system of glasses/wine/water at the beginning of a round of pouring and pouring back is determined by the total amount of wine in the first glass. Suppose at the beginning of some round, the first glass contains \(w\) pints of wine, \(0 \leq w \leq 1\) and \(1 - w\) pints of water. The second glass contains the rest of the wine and water.

Pouring 1/3 pint from the first glass to the second leaves 2/3 pints of liquid and \((2/3)w\) wine in the first glass, and 4/3 pints of liquid and \(1 - (2/3)w\) wine in the second glass. Pouring 1/3 pint back from the second into the first transfers a proportion of \((1/3)/(4/3)\) of the wine in the second glass into the first. So the round completes with both glasses containing a pint of liquid, and the first glass containing

\[
(2/3)w + (1/4)(1 - (2/3)w) = 1/4 + w/2
\]

pints of wine. After one more round, the first glass contains

\[
1/4 + (1/4 + w/2)/2 = 1/4 + 1/8 + w/2^2
\]

pints of wine, and after \(n\) more rounds

\[
w/2^n + \sum_{i=1}^{n} (1/2)^i + 1 = w/2^n + (1/2)^i \sum_{i=1}^{n} (1/2)^i = w/2^n + (1/2)(-1 + \sum_{i=0}^{n} (1/2)^i) = w/2^n + (1/2)(-1 + (1 - (1/2)^{n+1})/(1 - 1/2)) = w/2^n - 1/2 + 1 - (1/2)^{n+1} = w/2^n + 1/2 - (1/2)^{n+1}.
\]
Since $w = 0$ initially, the pints of wine in the first glass after $n$ rounds is
\[ \frac{1}{2} - (1/2)^{n+1}. \]

(b) What is the limit of the amount of wine in each glass as $n$ approaches infinity?

**Solution.** The limiting amount of wine in the first glass approaches $1/2$ from below as $n$ approaches infinity. In fact, it approaches $1/2$ no matter how the wine was initially distributed. This of course is what you would expect: after a thorough mixing the glasses should contain essentially the same amount of wine.

**Problem 2.**

An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine $d$ days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to make multiple trips carrying up to a gallon each time to create water caches out in the desert.

For example, if the shrine were $2/3$ of a day’s walk into the desert, then she could recover the Holy Grail after two days using the following strategy. She leaves the oasis with 1 gallon of water, travels $1/3$ day into the desert, caches $1/3$ gallon, and then walks back to the oasis—arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks $1/3$ day into the desert, tops off her water supply by taking the $1/3$ gallon in her cache, walks the remaining $1/3$ day to the shrine, grabs the Holy Grail, and then walks for $2/3$ of a day back to the oasis—again arriving with no water to spare.

But what if the shrine were located farther away?

(a) What is the most distant point that the explorer can reach and then return to the oasis, with no water precached in the desert, if she takes a total of only 1 gallon from the oasis?

**COMMENTS:**
- CP_holy_grail
- from: S09.cp9m
- has a commented out part

**keywords** = [harmonic_numbers]

**Solution.** At best she can walk $1/2$ day into the desert and then walk back.

(b) What is the most distant point the explorer can reach and still return to the oasis if she takes a total of only 2 gallons from the oasis? No proof is required; just do the best you can.

**Solution.** The explorer walks $1/4$ day into the desert, drops $1/2$ gallon, then walks home. Next, she walks $1/4$ day into the desert, picks up $1/4$ gallon from her cache, walks an additional $1/2$ day out and back, then picks up another $1/4$ gallon from her cache and walks home. Thus, her maximum distance from the oasis is $3/4$ of a day’s walk.

(c) The explorer will travel using a recursive strategy to go far into the desert and back, drawing a total of $n$ gallons of water from the oasis. Her strategy is to build up a cache of $n - 1$ gallons, plus enough to get home, a certain fraction of a day’s distance into the desert. On the last delivery to the cache, instead
of returning home, she proceeds recursively with her \( n - 1 \) gallon strategy to go farther into the desert and return to the cache. At this point, the cache has just enough water left to get her home.

Prove that with \( n \) gallons of water, this strategy will get her \( H_n/2 \) days into the desert and back, where \( H_n \) is the \( n \)th Harmonic number:

\[
H_n := \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.
\]

Conclude that she can reach the shrine, however far it is from the oasis.

**Solution.** To build up the first cache of \( n - 1 \) gallons, she should make \( n \) trips \( 1/(2n) \) days into the desert, dropping off \( (n - 1)/n \) gallons each time. Before she leaves the cache for the last time, she has \( n - 1 \) gallons plus enough for the walk home. Then she applies her \( (n - 1) \)-day strategy. So letting \( D_n \) be her maximum distance into the desert and back, we have

\[
D_n = \frac{1}{2n} + D_{n-1}.
\]

So

\[
D_n = \frac{1}{2n} + \frac{1}{2(n-1)} + \frac{1}{2(n-2)} + \cdots + \frac{1}{2} + \frac{1}{2 \cdot 1} = \frac{1}{2} \left( \frac{1}{n} + \frac{1}{(n-1)} + \frac{1}{(n-2)} + \cdots + \frac{1}{2} + \frac{1}{1} \right) = \frac{H_n}{2}.
\]

**Problem 3.**

Let \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) be a weakly decreasing function. Define

\[
S := \sum_{i=1}^{n} f(i)
\]

and

\[
I := \int_{1}^{n} f(x) \, dx.
\]

Prove that

\[
I + f(n) \leq S \leq I + f(1).
\]

(Proof by very clear picture is OK.)

**COMMENTS:**
• CP_sum_bound_by_integral
• Theorem: “brefweak_increasing_sum_bound” of the text

keywords = [ integral sum bound weakly_decreasing ]

Solution. Comparing the shaded regions in Figures 1(a) and 1(b),
\[ S \leq I + f(1). \]
Similarly, comparing the shaded regions in Figures 1(a) and 1(c),
\[ S \geq I + f(n). \]

Problem 4.
Sammy the Shark is a financial service provider who offers loans on the following terms.

• Sammy loans a client \( m \) dollars in the morning. This puts the client \( m \) dollars in debt to Sammy.

• Each evening, Sammy first charges a service fee which increases the client’s debt by \( f \) dollars, and then Sammy charges interest, which multiplies the debt by a factor of \( p \). For example, Sammy might charge a “modest” ten cent service fee and 1% interest rate per day, and then \( f \) would be 0.1 and \( p \) would be 1.01.

(a) What is the client’s debt at the end of the first day?

COMMENTS:

• PS_Sammy_the_shark
• from S07, pset7

keywords = [ asymptotics geometric_sum geometric_series ]

Solution. At the end of the first day, the client owes Sammy \( (m + f)p = mp + fp \) dollars.

(b) What is the client’s debt at the end of the second day?

Solution.

\[ ((m + f)p + f)p = mp^2 + fp^2 + fp \]

(c) Write a formula for the client’s debt after \( d \) days and find an equivalent closed form.

Solution. The client’s debt after three days is

\[ (((m + f)p + f)p + f)p = mp^3 + fp^3 + fp^2 + fp. \]

Generalizing from this pattern, the client owes

\[ mp^d + \sum_{k=1}^{d} fp^k \]
Figure 1  The area of the shaded region in (a) is $S = \sum_{i=1}^{n} f(i)$. The area in the shaded regions in (b) and (c) is $I = \int_{1}^{n} f(x) \, dx$. 
dollars after $d$ days. Applying the formula for a geometric sum gives:

$$mp^d + f \cdot \left( \frac{p^{d+1} - 1}{p - 1} - 1 \right)$$

(d) If you borrowed $10 from Sammy for a year, how much would you owe him?

**STAFF NOTE:** Calculator use expected.

**Solution.** $749.35$

```
(define (sammy m p d f)
  (+ (* m (expt p d))
      (* f (- (/ (- (expt p (+ d 1)) 1) (- p 1)) 1))))
```

```
(sammy 10 1.01 365 .1)
```

;Value: 749.3470300910349

**Supplemental problem**

**Problem 5.**

You’ve seen this neat trick for evaluating a geometric sum:

$$S = 1 + z + z^2 + \ldots + z^n$$

$$zS = z + z^2 + \ldots + z^n + z^{n+1}$$

$$S - zS = 1 - z^{n+1}$$

$$S = \frac{1 - z^{n+1}}{1 - z} \quad (\text{where } z \neq 1)$$

Use the same approach to find a closed-form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \ldots + nz^n$$

**COMMENTS:**

- CP_neat_trick_for_geometric_sum
- from: S09.cp9m, S05.rec10

**keywords** = [ series closed_form geometric_sum geometric_series ]

**Solution.**

$$zT = 1z^2 + 2z^3 + 3z^4 + \ldots + nz^{n+1}$$

$$T - zT = z + z^2 + z^3 + \ldots + z^n - nz^{n+1}$$

$$= \frac{1 - z^{n+1}}{1 - z} - 1 - nz^{n+1} \quad (\text{where } z \neq 1)$$

$$T = \frac{1 - z^{n+1}}{(1 - z)^2} - \frac{1 + nz^{n+1}}{1 - z} \quad (\text{where } z \neq 1)$$