Staff Solutions to In-Class Problems Week 10, Fri.

STAFF NOTE: Asymptotics, Ch. 14.7 (14.6 skipped)

Problem 1.
Recall that for functions $f, g$ on $\mathbb{N}$, $f = O(g)$ iff
\[ \exists c \in \mathbb{N} \exists n_0 \in \mathbb{N} \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|. \] (1)

For each pair of functions below, determine whether $f = O(g)$ and whether $g = O(f)$. In cases where one function is $O()$ of the other, indicate the smallest nonnegative integer $c$ and for that smallest $c$, the smallest corresponding nonnegative integer $n_0$ ensuring that condition (1) applies.

(a) $f(n) = n^2, g(n) = 3n$.

COMMENTS:

- CP_big_oh_practice
- from: S09.cp9t
- from: F02.quiz2
- has a lot of commented out material

keywords = [asymptotics big_oh]

Solution. ($f = O(g)$:) NO, because $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$.

Solution. ($g = O(f)$:) YES, with $c = 1, n_0 = 3$, which works because $3^2 = 9, 3 \cdot 3 = 9$.

(b) $f(n) = (3n - 7)/(n + 4), g(n) = 4$

Solution. ($f = O(g)$:) YES, with $c = 1, n_0 = 0$ (because $|f(n)| < 3$).

Solution. ($g = O(f)$:) NO, because $f(2n) = 1$, which rules out $g = O(f)$ since $g = \Theta(n)$.

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Problem 2.

(a) Indicate which of the following asymptotic relations below on the set of nonnegative real-valued functions are equivalence relations (E), strict partial orders (S), weak partial orders (W), or none of the above (N).

- \( f \sim g \), the “asymptotically equal” relation.
- \( f = o(g) \), the “little Oh” relation.
- \( f = O(g) \), the “big Oh” relation.
- \( f = \Theta(g) \), the “Theta” relation.

**COMMENTS:**
- **FP_asymptotics_define_functions**
- **from:** S09final.prob7, S07.MQ-2/28-1
- **Adapted by Jodyann F09**
- **partc from FP_multiple_choice adapted by Tom Brown**
- **first part edited by ARM 12/18/11**
- **2nd part added ARM 10/30/13**

**keywords = [ asymptotics little oh big oh Theta asymptotically_equal partial_order equivalence_relation implies ]**

Solution. E

\( f = o(g) \) is a strict partial order (S).

Solution. S

\( f = O(g) \) is a weak partial order (W).

Solution. N because it is neither symmetric nor antisymmetric.

Solution. E

\( f = \Theta(g) \) is an equivalence relation (E).

Solution. S

(b) Indicate the implications among the assertions in part (a). For example,

\[ f = o(g) \text{ implies } f = O(g). \]

Solution.

\[ f \sim g \text{ implies } f = \Theta(g) \text{ and } f = O(g) \text{ because } \lim(f/g) = 1 \text{ implies } \lim(f/g) = \lim(g/f) = 1 < \infty. \]

\[ f = o(g) \text{ implies } f = O(g) \text{ and } \text{not}(g = O(f)) \text{ because } \lim(f/g) = 0 \text{ implies } \lim(f/g) < \infty \text{ and } \lim(g/f) = \infty. \]

\[ f = \Theta(g) \text{ implies } f = O(g) \text{ (by definition of } \Theta). \]

It is not true that \( f = O(g) \text{ and } \text{not}(g = O(f)) \text{ implies } f = o(g). \) For example, let \( g(n) ::= n \) and \( f ::= g \cdot c_{\text{even}} \) where

\[ c_{\text{even}}(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{otherwise}. \end{cases} \]
Problem 3.

False Claim.

\[ 2^n = O(1). \]  

(2)

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.

Bogus proof. The proof is by induction on \( n \) where the induction hypothesis \( P(n) \) is the assertion (2).

base case: \( P(0) \) holds trivially.

inductive step: We may assume \( P(n) \), so there is a constant \( c > 0 \) such that \( 2^n \leq c \cdot 1 \). Therefore,

\[ 2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1. \]

which implies that \( 2^{n+1} = O(1) \). That is, \( P(n+1) \) holds, which completes the proof of the inductive step.

We conclude by induction that \( 2^n = O(1) \) for all \( n \). That is, the exponential function is bounded by a constant.

COMMENTS:

- CP_bogus_asymptotics_proof
- formerly CP_false_asymptotics_proof
- from: S09.cp9t

keywords = [ asymptotics induction false proof ]

Solution. The mistake in proof hinges on a misinterpretation of equation (2). To begin with, asymptotic relations are relations between functions. When we write \( O(1) \), we really mean “big-Oh of the constant function whose value is 1.” That is, if we define \( c_a \) to be the constant function equal to \( a \):

\[ c_a(x) := a, \quad \text{for all } x. \]

we should really have phrased (2) as:

\[ 2^n = O(c_1). \]  

(3)

But we still have the same issue with \( 2^n \) in (3). Does it refer to a constant that is a power of two, or does it refer to the exponential function? Now (3) is intended to refer to the exponential function. That is, it means

\[ \exp = O(c_1). \]  

(4)

where \( \exp \) is the function given by

\[ \exp(n) := 2^n. \]

This intended interpretation (4) is false, but the bogus proof claims to verify it.

The blunder is in misreading (3) as though it meant

\[ \forall n \in \mathbb{N}. \quad c_{2^n} = O(c_1). \]  

(5)

Assertion (5) is true, but uninteresting since all positive constant functions are \( O() \) of each other. This is not what not what (2) was intended to mean.

So the mistake in the bogus proof is in its misinterpretation of (2) as (5). The bogus proof then is a correct silly proof by an unnecessary induction of the true uninteresting assertion (5). Then in the last line, the bogus proof switches from the misinterpretation (5) to the intended interpretation (4).

It would be reasonable to say that the exact place where the bogus proof goes wrong is in its first line, where it defines \( P(n) \) based on misinterpretation (5). But we would describe this as a (very serious) strategic mistake, but not yet a specific mathematical mistake because the induction proof is correct. The exact place where the bogus proof makes a mathematical mistake is in its last line, when it switches from the misinterpretation (5) and mistakenly claims to have proved the false assertion (4).
Supplemental problems

Problem 4.
Assign true or false for each statement and prove it.

- \( n^2 \sim n^2 + n \)
- \( 3^n = O(2^n) \)
- \( n^{\sin(n\pi/2)+1} = o(n^2) \)
- \( n = \Theta\left(\frac{3n^3}{(n+1)(n-1)}\right) \)

COMMENTS:
- CP_asymptotic_true_false
- from S10
- edited ARM 10/30/13

keywords = [ asymptotic big_Oh little_oh Theta ]

Solution. The 1st and 4th statements are true.

- \( \frac{n^2+n}{n^2} = \frac{n^2}{n^2} + \frac{n}{n^2} = 1 + \frac{1}{n} \), so as \( n \) approaches infinity, the ratio approaches \( 1 + 0 = 1 \). Therefore the two expressions are asymptotically equal.

- \( \lim_{n \to \infty} \frac{2^n}{3^n} = \lim_{n \to \infty} (2/3)^n = 0 \), so \( 2^n = o(3^n) \) and therefore NOT \( (3^n = O(2^n)) \).

- The left side never exceeds \( n^2 \), but when \( n = 1, 5, 9, 13, \ldots \), the left side is equal to \( n^2 \), and so is not \( o(n^2) \).

- \( \lim_{n \to \infty} \frac{n}{\frac{3n^3}{(n+1)(n-1)}} = \lim_{n \to \infty} \frac{n(n+1)(n-1)}{3n^3} = \frac{1}{3} \),

so \( n = O \left( \frac{3n^3}{(n+1)(n-1)} \right) \). Similarly,

- \( \lim_{n \to \infty} \frac{\frac{3n^3}{(n+1)(n-1)}}{n} = 3 \),

so \( \frac{3n^3}{(n+1)(n-1)} = O(n) \). Because the two expressions are big-O of each other, \( n = \Theta \left( \frac{3n^3}{(n+1)(n-1)} \right) \).

Problem 5.
Give an elementary proof (without appealing to Stirling’s formula) that \( \log(n!) = \Theta(n \log n) \).

COMMENTS:
- CP_Theta_log_n_factorial
Solution. One elementary proof goes as follows:

First,

$$\log(n!) = \sum_{i=1}^{n} \log i < \sum_{i=1}^{n} \log n = n \log n.$$

On the other hand,

$$\log(n!) = \sum_{i=1}^{n} \log i > \sum_{i=\lceil (n+1)/2 \rceil}^{n} \log i$$

$$> \sum_{i=\lceil (n+1)/2 \rceil}^{n} \log(n/2) > \frac{n}{2} \cdot \log(n/2)$$

$$= \frac{n((\log n) - 1)}{2} = \frac{n \log n}{2} - \frac{n}{2}$$

$$> \frac{n \log n}{2} - \frac{n \log n}{6}$$

$$= \frac{1}{3} \cdot n \log n.$$

Therefore, $$(1/3)n \log n < \log(n!) < n \log n$$ for $n > 8$, proving that $\log(n!) = \Theta(n \log n)$. □