Staff Solutions to Problem Set 6

Reading:

- Chapter 12. Simple Graphs omitting Section 10.7.
- Chapter 14. Asymptotics through Section 14.5.

Problem 1.
Scholars through the ages have identified twenty fundamental human virtues: honesty, generosity, loyalty, prudence, completing the weekly course reading-response, etc. At the beginning of the term, every student in Math for Computer Science possessed exactly eight of these virtues. Furthermore, every student was unique; that is, no two students possessed exactly the same set of virtues. The Math for Computer Science course staff must select one additional virtue to impart to each student by the end of the term. Prove that there is a way to select an additional virtue for each student so that every student is unique at the end of the term as well.

Suggestion: Use Hall’s theorem. Try various interpretations for the vertices on the left and right sides of your bipartite graph.

Solution. Construct a bipartite graph $G$ as follows. The vertices on on the left are all students and the vertices on the right are all subsets of nine virtues. There is an edge between a student and a set of 9 virtues if the student already has 8 of those virtues.

Each vertex on the left has degree 12, since each student can learn one of 12 additional virtues. The vertices on the right have degree at most 9, since each set of 9 virtues has only 9 subsets of size 8. So this bipartite graph is degree-constrained, and therefore, by Lemma 12.5.6, there is a matching for the students. Thus, if each student is taught the additional virtue in the set of 9 virtues with whom he or she is matched, then every student is unique at the end of the term.

Problem 2.
Determine which among the four graphs pictured in Figure 1 are isomorphic. For each pair of isomorphic graphs, describe an isomorphism between them. For each pair of graphs that are not isomorphic, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, prove that it is indeed preserved under isomorphism (you only need prove one of them).

Solution. $G_1$ and $G_4$ are isomorphic. In particular, the function $f : V(G_1) \to V(G_4)$ is an isomorphism, where

$$
\begin{align*}
f(1) &= 1 \\
f(2) &= 2 \\
f(3) &= 3 \\
f(4) &= 8 \\
f(5) &= 9 \\
f(6) &= 10 \\
f(7) &= 4 \\
f(8) &= 5 \\
f(9) &= 6 \\
f(10) &= 7
\end{align*}
$$

$G_1$ and $G_2$ are not isomorphic to $G_3$: $G_3$ has a vertex of degree four and neither $G_1$ nor $G_2$ has one. $G_1$ and $G_2$ are not isomorphic: $G_2$ has a cycle of length four and $G_1$ does not.
There are many examples of properties preserved under graph isomorphism noted in Chapter 12; for example, number of vertices and edges, vertex degrees, size of cycles and connectedness. See Problem 12.8 for a formal proof that isomorphisms preserve vertex degrees.

**Problem 3.**
This problem generalizes the result proved Theorem 12.6.3 that any graph with maximum degree at most \( w \) is \((w + 1)\)-colorable.

A simple graph, \( G \), is said to have **width** \( w \) iff its vertices can be arranged in a sequence such that each vertex is adjacent to at most \( w \) vertices that precede it in the sequence. If the degree of every vertex is at most \( w \), then the graph obviously has width at most \( w \)—just list the vertices in any order.

(a) Prove that every graph with width at most \( w \) is \((w + 1)\)-colorable.

**Solution.** We use induction on \( n \), the number of vertices. Let \( P(n) \) be the proposition that for all \( w \), every \( n \)-vertex graph with width \( w \) is \((w + 1)\)-colorable.

**Base case:** \((n = 1)\) Every graph with 1 vertex has width 0 and is \( 0 + 1 = 1 \) colorable. Therefore, \( P(1) \) is true.

**Inductive step:** Now we assume \( P(n) \) in order to prove \( P(n + 1) \). Let \( G \) be an \((n + 1)\)-vertex graph with width at most \( w \). This means that the \( n + 1 \) vertices can be arranged in a sequence, \( S \), such that each vertex is connected to at most \( w \) preceding vertices. Removing the last vertex, \( v \), and all edges incident to it gives a subgraph \( G' \) with \( n \) vertices. The subgraph \( G' \) also has width at most \( w \), since the sequence \( S \) with its last vertex removed is a sequence of all the vertices of \( G' \) with each vertex adjacent to exactly the same previous
vertices. So by Induction Hypothesis, $G'$ is $(w + 1)$-colorable. But any $(w + 1)$-coloring of $G'$ can be extended to a $(w + 1)$-coloring of $G$ by assigning a color to $v$ that differs from the colors of its adjacent vertices. Since there are at most $w$ colors among the $w$ vertices adjacent to $v$, there will always be a different one of the $w + 1$ colors to assign to $v$. So $G$ is $(w + 1)$-colorable, which proves $P(n + 1)$. This completes the proof of the Induction step.

The result now follows for all $G$ by the Principle of Induction.

(b) Describe a 2-colorable graph with minimum width $n$.

**STAFF NOTE:** Hint: The complete bipartite graph $K_{n,n}$.

**Solution.** An example would be the complete bipartite graph $K_{n,n}$. Since it is bipartite, it is 2-colorable.

But every vertex has degree $n$ and therefore has minimum width $n$, since the last vertex in any list of the vertices is adjacent to $n$ preceding vertices.

(c) Prove that the average degree of a graph of width $w$ is at most $2w$.

**Solution.** If we line up the vertices, we can define the backdegree of a vertex to be the number of preceding vertices it is adjacent to. The sum of the back degrees equals the number, $e$, of edges. Since there is a sequence in which all the back degrees are at most $w$, the total number of edges is at most $w$ times the number, $n$, of vertices. But by the Handshaking Lemma, the sum of all the degrees is $2e$, so the average degree is $2e/n \leq 2wn/n = 2w$.

(d) Describe an example of a graph with 100 vertices, width 3, but average degree more than 5.

*Hint:* Don’t get stuck on this; if you don’t see it after five minutes, ask for a hint.

**Solution.** The hint is to line up the 100 vertices and have each vertex be adjacent to the 3 immediately preceding vertices, if any. By definition of width, this graph has width 3. All vertices other than the first three are now adjacent to three preceding vertices, and all vertices except the last three are also adjacent to the three following vertices. So vertices 4 through 97 all have degree 6; this alone ensures that the average degree is at least $6 \cdot 94/100 = 5.76$.

**Problem 4.**

Prove Corollary 12.9.12: If all edges in a finite weighted graph have distinct weights, then the graph has a unique MST.

*Hint:* Suppose $M$ and $N$ were different MST’s of the same graph. Let $e$ be the smallest edge in one and not the other, say $e \in M − N$, and observe that $N + e$ must have a cycle.

**Solution.** Assume for the sake of contradiction that $M$ and $N$ were different MST’s of the same graph. Let $e$ be a minimum weight edge as in the hint.

Since $N$ is a spanning tree, we know that $N + e$ is connected, and it has too many edges to be a tree, so $N + e$ has a cycle. Since $M$ has no cycles, the cycle in $N + e$ cannot consist solely of edges from $M$. So there must be an edge $g \in N − M$ on the cycle, and we know that $w(g)$ must be larger than $w(e)$ by definition of $e$. Removing $g$ from $N + e$ leaves a connected graph with the same number of nodes and edges as $N$, so $N + e − g$ must be a spanning tree. But $N + e − g$ weighs $w(g) − w(e)$ less than $N$, contradicting the fact that $N$ is a minimum weight spanning tree.
Problem 5.
Is a Harvard degree really worth more than an MIT degree? Let us say that a person with a Harvard degree starts with $40,000 and gets a $20,000 raise every year after graduation, whereas a person with an MIT degree starts with $30,000, but gets a 20% raise every year. Assume inflation is a fixed 8% every year. That is, $1.08 a year from now is worth $1.00 today.

(a) How much is a Harvard degree worth today if the holder will work for \( n \) years following graduation?

(b) How much is an MIT degree worth in this case?

(c) If you plan to retire after twenty years, which degree would be worth more?

Solution. One dollar after year \( i \) is worth \( r^i \) in today’s currency, where

\[
r = \frac{1}{1.08} = 0.925925925 \ldots
\]

So

\[
\text{Hvd}_n = \sum_{i=0}^{n} (40000 + 20000i) r^i
\]

\[
= 40000 \sum_{i=0}^{n} r^i + 20000 \sum_{i=0}^{n} i r^i,
\]

\[
\text{MIT}_n = \sum_{i=0}^{n} 1.2^i r^i
\]

\[
= 30000 \sum_{i=0}^{n} (1.2)^i
\]

But

\[
\sum_{i=0}^{n} i r^i = \frac{r - (n + 1)r^{n+1} + nr^{n+2}}{(1 - r)^2},
\]

so

\[
\text{Hvd}_n = 40000 \frac{(1 - r^{n+1})}{1 - r} + 20000 \frac{(r - (n + 1)r^{n+1} + nr^{n+2})}{(1 - r)^2}
\]

\[
= \frac{20000(2(1 - r^{n+1} - r + r^{n+2}) + r - (n + 1)r^{n+1} + nr^{n+2})}{(1 - r)^2}
\]

\[
= \frac{20000(2 - r - (n + 3)r^{n+1} + (n + 2)r^{n+2})}{(1 - r)^2}
\]

\[
\text{MIT}_n = \frac{30000(1 - (1.2)^{n+1})}{1 - 1.2r}
\]

and for \( n = 20 \),

\[
\text{Hvd}_{20} = \frac{20000(2 - r - 23r^{21} + 22r^{22})}{(1 - r)^2} = 2,010,885
\]

\[
\text{MIT}_{20} = \frac{30000(1 - (1.2)^{21})}{1 - 1.2r} = 2,197,579.
\]

so the MIT degree is more valuable! (But we knew that already.)
Problem 6.
Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

\[
\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}
\]

Solution. Let’s first try standard bounds:

\[
\int_0^\infty \frac{1}{2(x+3)^2} \, dx \leq \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \leq \int_0^\infty \frac{1}{(2x+1)^2} \, dx
\]

Evaluating the integrals gives:

\[
-\frac{1}{2(2x+3)} \Bigg|_0^\infty \leq \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \leq -\frac{1}{2(2x+1)} \Bigg|_0^\infty
\]

\[
\frac{1}{6} \leq \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \leq \frac{1}{2}
\]

These bounds are too far apart, so let’s sum the first couple terms explicitly and bound the rest with integrals.

\[
\frac{1}{3^2} + \frac{1}{5^2} + \int_2^\infty \frac{1}{(2x+3)^2} \, dx \leq \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \leq \frac{1}{3^2} + \frac{1}{5^2} + \int_2^\infty \frac{1}{(2x+1)^2} \, dx
\]

Integration now gives:

\[
\frac{1}{3^2} + \frac{1}{5^2} + \left(-\frac{1}{2(2x+3)} \Bigg|_2^\infty \right) \leq \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \leq \frac{1}{3^2} + \frac{1}{5^2} + \left(-\frac{1}{2(2x+1)} \Bigg|_2^\infty \right)
\]

\[
\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{14} \leq \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \leq \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{10}
\]

Now we have bounds that differ by \(1/10 - 1/14 < 1/10 = 0.1\).