Staff Solutions to Midterm Exam April 26

Remember, part credit is more easily earned when an explanation is included.

STAFF NOTE: ARM used for Taylor Shaw Problem 1:

part(a) “40 · 41” gets 1 pt,
part(e) “2” get 0pts,
part(i) “(41^10)” gets 0pts.

Problem 1 (Graphs & Counting) (20 points).
Answer the following questions about finite simple graphs. You may answer with formulas involving exponents, binomial coefficients, and factorials.

(a) How many edges are there in the complete graph $K_{41}$?

Solution.

\[
\binom{41}{2} = 820.
\]

(b) How many edges are there in a spanning tree of $K_{41}$?

Solution. 40.

(c) What is the chromatic number $\chi(K_{41})$?

Solution. 41.

(d) What is the chromatic number $\chi(C_{41})$, of the cycle of length 41?

Solution. 3.

(e) Let $H$ be the graph in Figure 1. How many distinct isomorphisms are there from $H$ to $H$?

Solution. 4.

Figure 1  The graph $H$. 
(f) A graph $G$ is created by adding a single edge to a tree with 41 vertices. How many cycles does $G$ have?

Solution. Exactly 1.

(g) What is the smallest number of leaves possible in a tree with 41 vertices?

Solution. 2.

(h) What is the largest number of leaves possible in a tree with 41 vertices?

Solution. 40.

(i) How many length-10 paths are there in $K_{41}$?

Solution. \[
\frac{41!}{30!}\]

There is a bijection between length-10 paths and length-11 sequences of vertices.

(j) Let $s$ be the number of length-10 paths in $K_{41}$—that is, $s$ is the correct answer to part (i). In terms of $s$, how many length-11 cycles are in $K_{41}$?

Hint: For vertices $a, b, c, d, e$, the sequences $abcde, bcdea, and edcba$ all describe the same length-5 cycle, for example.

Solution. \[
\frac{s}{11 \cdot 2}\]

A cycle is determined by the sequence of 11 vertices in it, but since a cycle has no “first” or “last” vertex, 11 different sequences determine the same cycle in clockwise order, and another 11—the reversals of the first 11—determine it in counterclockwise order. So the number of cycles equals the number of sequences, namely $s = 41!/30!$, divided by $11 \cdot 2$.

Problem 2 (Asymptotic Sum) (15 points).

Prove that

$$\sum_{k=1}^{n} k^6 = \Theta(n^7).$$

Solution. Let $S_n := \sum_{k=1}^{n} k^6$. 

One approach is to use the Integral Method, Theorem 14.3.2. Since $f(k) := k^6$ is an increasing function, we have

\[
\frac{n^7}{7} \leq 1^6 + \frac{n^7}{7} - \frac{1^7}{7} \\
= f(1) + \int_1^n x^6 \, dx \\
\leq S_n \quad \text{(Theorem 14.3.2)}
\]

\[
\leq f(n) + \int_1^n x^6 \, dx \\
= n^6 + \frac{n^7}{7} - \frac{1^7}{7} \\
= O(n^7).
\]

So $n^7 = O(S_n)$, and $S_n = O(n^7)$. That is, $n^7 = \Theta(S_n)$.

An alternative approach not using the Integral Method goes as follows. There are $n$ terms in $S_n$ and each term is at most $n^6$, so $S_n \leq n \cdot n^6 = n^7 = O(n^7)$. So $S_n = O(n^7)$.

On the other hand, at least $(n - 1)/2$ of the terms are as large as $[(n - 1)/2]^6$, so

\[
S_n \geq ((n - 1)/2) \cdot [(n - 1)/2]^6 \\
= [(n - 1)/2]^7 \\
\geq (n/3)^7
\]

for $n > 3$, so $n^7 \leq 3^7 \cdot S_n$. In other words, $n^7 = O(S_n)$.

\[\blacksquare\]

**Problem 3 (Pigeonholing) (15 points).**

Show that in any set of 100 integers, there must be fifteen such that the difference of any two among the fifteen is a multiple of 7.

**Solution.** Let the 100 integers be pigeons and the numbers $[0..6]$ be the holes. Map a pigeon to its remainder on division by 7. By the generalized pigeonhole principle, there must be $\lceil 100/7 \rceil = 15$ pigeons in one hole. Since these 15 pigeons have the same remainder, the difference of any two is divisible by 7.

**STAFF NOTE:** ARM used for Taylor Shaw Problem 4

part(a) 7pts, part(b) 9pts, part(c) 4 pts.

\[\blacksquare\]

**Problem 4 (Multinomial Coefficients & Congruence) (20 points).**

Let $p$ be a **prime number**.

(a) Explain why the multinomial coefficient

\[
\binom{p}{k_1, k_2, \ldots, k_n}
\]

is divisible by $p$ if all the $k_i$’s are nonnegative integers less than $p$.

**Solution.** The multinomial coefficient is an integer equal to the quotient of $p!$ divided by the product $k_1!, k_2!, \ldots, k_n!$. If all the $k_i$’s are less than $p$, then none of the denominator terms divides the numerator, $p$, and so the multinomial coefficient is divisible by $p$. 

\[\blacksquare\]
(b) Conclude from part (a) that
\[(x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + x_2^p + \cdots + x_n^p \pmod{p}. \tag{1}\]
(Do not prove this using Fermat’s “little” Theorem. The point of this problem is to offer an independent proof of Fermat’s theorem.)

**Solution.** By the Multinomial Theorem 15.6.5, \((x_1 + x_2 + \cdots + x_n)^p\) is a sum of monomials in \(x_1, \ldots, x_n\) whose coefficients are of the form given in part(a).

Since the sum of the \(k_i\)’s is \(p\), the only coefficients not divisible by \(p\) are the coefficients where some \(k_i = p\) and all the other \(k_j\)’s are zero. That is, the only coefficients not \(\equiv 0 \pmod{p}\) are the coefficients
\[
\binom{p}{0, 0, \ldots, 0, p, 0, \ldots, 0} = 1
\]
of the monomials \(x_i^p\). The congruence (1) follows immediately. ■

(c) Explain how (1) immediately proves Fermat’s Little Theorem:
\[n^{p-1} \equiv 1 \pmod{p}\]
when \(n\) is not a multiple of \(p\).

**Solution.** Let \(x_1 = x_2 = \cdots x_n = 1\). Then (1) implies \(n^p \equiv n \cdot 1^p = n \pmod{p}\). If \(n\) is not a multiple of \(p\), then we can then cancel \(n\) to get \(n^{p-1} \equiv 1 \pmod{p}\). ■

**Problem 5** (Bipartite Matching) (**10 points**).
Let \(G\) be a 2-colorable simple graph in which every vertex has the same positive degree. Explain why \(G\) has a perfect matching.

**Solution.** Since \(G\) is 2-colorable, it is bipartite with \(L(G)\) being the vertices of one color and \(R(G)\) the vertices of the other color. Moreover, since all vertices have the same degree, \(G\) is *degree-constrained*. Therefore \(G\) has a perfect matching by Theorem 12.5.6. ■

**Problem 6** (Card Counting) (**20 points**).
In a standard 52-card deck (13 ranks and 4 suits), a hand is a 5-card subset of the set of 52 cards. Express the answer to each part as a formula using factorial, binomial, or multinomial notation.

(a) Let \(H\) be the set of all hands. What is \(|H|\)?

**Solution.**
\[|H| = \binom{52}{5}\]
■

(b) Let \(H_{NP}\) be the set of all hands that include no pairs; that is, no two cards in the hand have the same rank. What is \(|H_{NP}|\)?
Solution.

\[ |H_{NP}| = \binom{13}{5} \binom{4}{1}^5 \]

(c) Let \( H_S \) be the set of all hands that are straights, that is, the ranks of the five cards are consecutive. The order of the ranks is \((A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A)\); note that \( A \) appears twice. What is \( |H_S| \)?

Solution.

\[ |H_S| = \binom{10}{1} \binom{4}{1}^5 \]

(d) Let \( H_F \) be the set of all hands that are flushes, that is, the suits of the five cards are identical. What is \( |H_F| \)?

Solution.

\[ |H_F| = \binom{4}{1} \binom{13}{5} \]

(e) Let \( H_{SF} \) be the set of all straight flush hands, that is, the hand is both a straight and a flush. What is \( |H_{SF}| \)?

Solution.

\[ |H_{SF}| = \binom{10}{1} \binom{4}{1} \]

(f) Let \( H_{HC} \) be the set of all high-card hands; that is, hands that do not include pairs, are not straights, and are not flushes. Write a formula for \( |H_{HC}| \) in terms of \( |H_{NP}|, |H_S|, |H_F|, |H_{SF}| \).

Solution.

\[ |H_{HC}| = |H_{NP}| - |H_S| - |H_F| + |H_{SF}|. \tag{2} \]

This follows because

\[ H_{NP} = H_S \cup H_F \cup H_{HC} \]

by definition of the poker hands. Also, \( H_{HC} \) is disjoint from \( H_S \) and \( H_F \). So by inclusion exclusion,

\[ |H_{NP}| = |H_{HC}| + |H_S| + |H_F| - 0 - 0 - |H_{SF}| + 0, \]

which immediately yields (2).