Staff Solutions to Midterm Exam February 23

Problem 1 (Well Ordered Stamps) (15 points).
Prove using the Well Ordering Principle that, using 6¢, 14¢, and 21¢ stamps, it is possible to make any amount of postage over 50¢. To save time, you may specify assume without proof that 50¢, 51¢, … 100¢ are all makeable, but you should clearly indicate which of these assumptions your proof depends on.

Solution. Proof. Assume to the contrary that some amount of postage of 50¢ or more is not makeable. So by WOP, there will be a least unmakeable amount \( m \geq 50 \). If we assume 50–55¢ is makeable, then we can conclude that \( m \geq 56 \). So \( m - 6 \geq 50 \) and therefore is makeable, because if \( m > k \geq 50 \), then \( k \) is makeable by definition of \( m \). Now since \( m - 6 \) is makeable, we can add a 6¢ stamp and make \( (m - 6) + 6 = m \), contradicting the fact that \( m \) is unmakeable. So there cannot be such a minimum \( m \), which proves that all amounts \( \geq 50 \)¢ are makeable.

Problem 2 (Irrational Contradiction) (20 points).
A familiar proof that \( \sqrt{7} \) is irrational depends on the fact that a certain equation among those below is unsatisfiable by integers \( a, b > 0 \). Note that More than one is unsatisfiable. Circle the equation that would appear in the proof, and explain why it is unsatisfiable. (Do not assume that \( \sqrt[3]{7} \) is irrational.)

i. \( a^2 = 7^2 + b^2 \)
ii. \( a^3 = 7^2 + b^3 \)
iii. \( a^2 = 7^2 b^2 \)
iv. \( a^3 = 7^2 b^3 \)
v. \( a^3 = 7^3 b^3 \)
vi. \( (ab)^3 = 7^2 \)

STAFF NOTE: 20pt problem, 2pts (that is, -18pt) for picking an irrelevant equation, say \( (ab)^3 = 7^2 \), and showing it is unsatisfiable.

Solution.
\[ a^3 = 7^2 b^3. \]

By the unique factorization theorem, the prime factorization of \( a^3 \) is obtained by repeating the prime factors of \( a \) three times. So every prime in the factorization of \( a^3 \) must appear three times. Likewise for \( b^3 \).

But the number of appearances of 7 in the factorization of the right hand side, \( 7^2 b^3 \), leaves a remainder of two when divided by three, and so will not equal the number of appearances of 7 in factorization of the left hand side, \( a^3 \).
Problem 3 (Propositional Connectives) (25 points).
Let \( P \) be a propositional variable.
(a) Show how to express \( \text{NOT}(P) \) using \( P \) and a selection from among the constant \textbf{True}, and the connectives XOR and AND.

**STAFF NOTE:** Rubric: (a) 6pts (b) 6pts (c) 13pts.

for (c) Statements that \( P \text{ XOR } P \equiv \text{False} \) and \( P \text{ AND } P \equiv P \) w/o explanation of how these equivalences explain the conclusion gets -5pts = 8 of 13. We’re looking for some reference to WOP as in the soln or some other proper explanation.

Solution.
\[
\text{NOT}(P) \equiv P \text{ XOR True}.
\]

(b) Explain why part (a) implies that every propositional formula is equivalent to one whose only connectives are XOR and AND, along with the constant True.

Solution. We know that every propositional formula is equivalent to one using only NOT, AND, and OR. Moreover by DeMorgan’s Law (Section 3.4.2) OR is expressible in terms of NOT and AND. So by expressing NOT according to part (a), every formula is equivalent to one that can be expressed using only XOR, AND, and True.

(c) The constant \textbf{True} is essential for part (b). This follows because every propositional formula using only \( P \), the connectives XOR and AND, and no constants—call this a “PXA-formula”—is equivalent to \( P \) or to \textbf{False}. Prove this claim.

Hint: Use WOP and look at the shortest PXA-formula that might not be equivalent to \( P \) or \textbf{False}.

Solution. Suppose there is a PXA-formula not equivalent to \( P \) or to \textbf{False}. By WOP, there will be a shortest such formula, \( F \).

Now \( F \) cannot consist of just the propositional variable \( P \), since \( P \) is equivalent to \( P \). Therefore, \( F \) must be of the form “\( G \text{ XOR } H \)” or “\( G \text{ AND } H \)” for some PXA-formulas \( G \) and \( H \). But since \( G \) and \( H \) are shorter than \( F \), they must each be equivalent to \( P \) or to \textbf{False}. This leads to the contradiction that \( F \) is equivalent to \( P \) or to \textbf{False}, since \( X \text{ AND } Y \) and \( X \text{ XOR } Y \) are equivalent to \( P \) or to \textbf{False} when \( X, Y \) take the values \( P \) and/or \textbf{False}.

Problem 4 (Domains of Discourse) (10 points).
For each of the logic formulas below, indicate the smallest domain in which it is true, among

\[ \mathbb{N}(\text{nonnegative integers}), \mathbb{Z}(\text{integers}), \mathbb{Q}(\text{rationals}), \mathbb{R}(\text{reals}), \mathbb{C}(\text{complex numbers}) \]

or write “\textit{none}” if it is not true in any of them. Do not include explanations.

i. \( \forall x \exists y. \ y = 3x \)

Solution. \( \mathbb{N} \)

ii. \( \forall x \exists y. \ 3y = x \)

Solution. \( \mathbb{Q} \)
iii. \( \forall x \exists y. \ y^2 = x \)

Solution. \( \mathbb{C} \)

iv. \( \forall x \exists y. \ y < x \)

Solution. \( \mathbb{Z} \)

v. \( \forall x \exists y. \ y^3 = x \)

Solution. \( \mathbb{R} \) Every real number does have a cube root.

vi. \( \forall x \neq 0. \exists y, z. \ y \neq z \land \ y^2 = x = z^2 \)

Solution. \( \mathbb{C} \) Every nonzero has 2 sqrts.

Problem 5 (A Set Identity) (20 points).
The set equation
\[
A \cap \overline{B} = \overline{A} \cup B
\]
follows from a certain equivalence between propositional formulas.

(a) What is the equivalence?

Solution. DeMorgan’s Law
\[
\text{NOT}(A \land B) = \text{NOT}(A) \lor \text{NOT}(B).
\]

(b) Show how to derive the equation from this equivalence.

Solution. We will prove the equality by showing that the left hand and right hand sets have the same members. That is, we will prove:
\[
x \in A \cap \overline{B} \text{ iff } x \in \overline{A} \lor \overline{B}.
\]

Proof. 
\[
x \in A \cap \overline{B}
\]
iff \( \text{NOT}(x \in A \cap B) \) \quad (def of set complement)
iff \( \text{NOT}(x \in A \land x \in B) \) \quad (def of \( \cap \))
iff \( \text{NOT}(x \in A) \lor \text{NOT}(x \in B) \) \quad (DeMorgan’s Law)
iff \( x \in \overline{A} \lor x \in \overline{B} \) \quad (def of set complement)
iff \( x \in \overline{A} \cup \overline{B} \) \quad (def of \( \cup \)).

Problem 6 (Counter Machines) (10 points).
Write a program for a counter machine with two counters \( R \) and \( S \) that adds 3 times the contents of \( R \) to the contents of \( S \), while setting the contents of \( R \) to 0. That is, the machine simulates the assignment statements...
\[ S := S + 3 \times R; \]
\[ R := 0 \]

Your program should only use the basic Counter Machine instructions \( T^+ \) (increment counter \( T \)), \( T^- \) (decrement counter \( T \) unless it contains 0), \( [T? \ m \ n] \) (if \( T \) contains 0, goto line number \( m \), otherwise goto line \( n \)) for any counter \( T \).

*Hint:* There is a six line program that needs no extra counters beyond \( R \) and \( S \). No penalty for longer correct programs using extra registers.

**Solution.**

1. \([R? \ \text{halt} \ 2]\), 2. \(R^-\), 3. \(S^+\), 4. \(S^+\), 5. \(S^+\), 6. \(\text{goto} \ 1\)
   which is shorthand for:

1. \([R? \ 7 \ 2]\), 2. \(R^-\), 3. \(S^+\), 4. \(S^+\), 5. \(S^+\), 6. \([S? \ 1 \ 1]\)

Even nicer: 1. \([R? \ 7 \ 2]\), 2. \(R^-\), 3. \(S^+\), 4. \(S^+\), 5. \(S^+\), 6. \([R? \ 7 \ 2]\)