Staff Solutions to 2nd Conflict Midterm Exam April 5

Problem 1 (GCD) (15 points).
Prove that
\[ \gcd(a^5, b^5) = \gcd(a, b)^5 \]
for all \( a, b \in \mathbb{Z} \).

Solution. Proof. Let \( v_k(n) \) be the largest power of \( k \) that divides \( n \), that is,
\[ v_k(n) := \max\{i \mid k^i \text{ divides } n\}. \]

Hence,
\[ v_k(mn) = v_k(m) + v_k(n). \]

By the Prime Factorization Theorem, we know that
\[ v_p(\gcd(m, n)) = \min(v_p(m), v_p(n)), \]
and so
\[ v_p(a^5) = 5v_p(a), \]
by (*).

Problem 2 (GCD/Congruence) (15 points).
Show that there is an integer \( x \) such that
\[ ax \equiv b \pmod{n} \]
iff
\[ \gcd(a, n) \mid b. \]

Solution. The proof follows almost immediately from the fact that the linear combinations of \( c, d \in \mathbb{Z} \) are the same as the multiples of \( \gcd(c, d) \).

\[ \exists x. \ ax \equiv b \pmod{n} \iff \exists x, \exists y. ax + ny = b \]
iff
\[ b \text{ is a linear combination of } a \text{ and } n \]
iff
\[ b \text{ is a multiple of } \gcd(a, n) \]
iff
\[ \gcd(a, n) \mid b. \]
Problem 3 (Partial Order/Equivalence) (15 points).
Prove that for any nonempty set $D$, there is a unique binary relation on $D$ that is both a weak partial order and also an equivalence relation.

Solution. The unique relation is the identity relation, $\text{Id}_D$. The identity relation is obviously an equivalence relation and is vacuously a weak partial order (in which no element is comparable to any other element).

So it is only necessary to show uniqueness, that is, if $R$ is an equivalence relation and a weak partial order on $D$, then $R = \text{Id}_D$. Since $R$ is obviously reflexive, we need only show that if $a R b$, then $a = b$.

So suppose $a R b$. Since $R$ is an equivalence relation, it is symmetric, and so $b R a$. Since $R$ is a weak partial order, it is antisymmetric, so from $a R b$ and $b R a$, we conclude that $a = b$, as required.

Problem 4 (Partial Orders/Scheduling) (20 points).
Answer the following questions about the powerset, $\text{pow}(\{1, 2, 3, 4\})$, partially ordered by the strict subset relation $\subset$.

(a) Give an example of a maximum length chain.

Solution.

$$\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}.$$  

(b) Give an example of an antichain of size 6.

Solution.

$$\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{1, 4\}.$$  

(c) Suppose the partial order describes scheduling constraints on 16 tasks. That is, if $A \subset B \subseteq \{1, 2, 3, 4\}$, then $A$ has to be completed before $B$ starts. What is the minimum number of processors needed to complete all the tasks in minimum parallel time? Prove it.

Solution. 4.

A minimum time schedule takes length of max chain steps, which by part (a) is 5. There is a unique minimum task, $\emptyset$, which must come first and a unique maximum task, $\{1, 2, 3, 4\}$, which must come last; this leaves 14 tasks which require at least $\lceil 14/k \rceil$ more parallel steps with $k$ processors. So for min time, we need $\lceil 14/k \rceil \leq 3$, which implies that $k \geq 4$. Moreover, there is a 4-processor schedule that takes 5 steps:

For example, a length-6 4-processor schedule is:

$$\emptyset$$
$$1, 2, 3, 4$$
$$12, 23, 13, 14$$
$$24, 34, 123$$
$$234, 124, 134$$
$$1234.$$  

1As usual, we assume each task requires one time unit to complete.
Problem 5 (Simple Graphs) (20 points).
The degree sequence of a simple graph $G$ with $n$ vertices is the length-$n$ sequence of the degrees of the vertices listed in weakly increasing order. For example, if $G$ is a 4-vertex tree, then its degree sequence is either $(1,1,1,3)$ or $(1,1,2,2)$.

Briefly explain why each of the following sequences is not a degree sequence of any connected simple graph.

(a) $(1, 2, 3, 4, 5, 6, 7)$

Solution. There are only 7 vertices, so the degree of any vertex is at most 6.

(b) $(0, 2, 2, 2, 2)$

Solution. There is a vertex with degree 0, and there is more than one vertex, so the graph is not connected.

(c) $(1, 3, 3, 4, 4, 4)$

Solution. By the Handshaking Lemma, the sum of degrees in any simple graph must be even, which is not true in this case since $1 + 3 + 3 + 4 + 4 + 4 = 19$.

(d) A sequence of $n$ integers whose sum is less than $2n - 2$.

Solution. There are too few edges. The sum of the degrees is twice the number of edges by the Handshaking Lemma 12.2.1. However, the number of edges in any $n$-vertex connected graph is at least $n - 1$ (Corollary 12.8.8). Therefore, the sum of the degrees must be at least $2n - 2$.

(e) $(1, 2, 3, 4, 4)$

Solution. There are five vertices, two of which have degree 4. So both degree-4 vertices have to be connected to all the other vertices. That means that the degree of every vertex is greater than one, which is violated in this case.

Problem 6 (Partial Orders/Chains) (15 points).
Let $R$ be a weak partial order on a set, $A$. Suppose $C$ is a finite chain.

that $C$ has a maximum element. *Hint: Induction on the size of $C$.*

Solution. As hinted, we give a proof by induction on the size of $C$.

Proof. The induction hypothesis is:

$$P(n) := \text{If } C \text{ is a chain of size } n, \text{ then } C \text{ has a maximum element.}$$

\[2\text{A set } C \text{ is a chain when it is nonempty, and all elements } c, d \in C \text{ are comparable. Elements } c \text{ and } d \text{ are comparable iff } [c R d \text{ or } d R c].\]
**Base case**: \((n = 1)\). The one element is \(C\) is the maximum (ands also minimum) element, bym definition of maximum.

**Induction step**: To prove \(P(n + 1)\) for \(n \geq 1\), let \(C_{n+1}\) be a chain of size \(n + 1\) and let \(x\) be an arbitrary element in \(C_{n+1}\). Then \(C_{n+1} - \{x\}\) is a chain of size \(n\), so it has a maximum element \(m\) by induction hypothesis. Now compare \(x\) and \(m\). If \(x \mathrel{R} m\), then \(m\) is the maximum element in \(C_{n+1}\). On the other hand, \(m \mathrel{R} x\), then (by transitivity of \(R\)), \(x\) is a maximum element of \(C_{n+1}\). In any case, \(C_{n+1}\) has a maximum element, which proves \(P(n + 1)\).  

\[\blacksquare\]