Staff Solutions to Final Exam

STAFF NOTE: (a),(b),(c) 5 pts each

Problem 1 (Stable matching) (15 points).
Four boys and four girls have the following preference rankings:

<table>
<thead>
<tr>
<th>Boy</th>
<th>Girl 1</th>
<th>Girl 2</th>
<th>Girl 3</th>
<th>Girl 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alfred</td>
<td>Helen</td>
<td>Emily</td>
<td>Fiona</td>
<td>Grace</td>
</tr>
<tr>
<td>Billy</td>
<td>Helen</td>
<td>Grace</td>
<td>Fiona</td>
<td>Emily</td>
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<tr>
<td>Calvin</td>
<td>Grace</td>
<td>Helen</td>
<td>Emily</td>
<td>Fiona</td>
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<tr>
<td>David</td>
<td>Helen</td>
<td>Grace</td>
<td>Emily</td>
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</tr>
<tr>
<td>Helen</td>
<td>Calvin</td>
<td>Billy</td>
<td>David</td>
<td>Alfred</td>
</tr>
</tbody>
</table>

(a) Explain why Grace and Calvin must be married in every stable matching.

Solution. Each is the other’s first choice, so they would be a rogue couple if they were not married to each other.

(b) Explain why it follows that Helen and Billy must also be married in every stable matching.

Solution. With Grace and Calvin matched, Helen and Billy are each other’s remaining first choice, so they must similarly be matched.

(c) Finally, explain why there is only one possible stable matching.

Solution. With the four above out of the way, Alfred and Emily are each other’s remaining first choice, so they must similarly be matched, which forces Fiona to marry David.

Problem 2 (Modular Inverse) (10 points).
Explain why 1059 does not have an inverse modulo 1412.

Solution. 1412 and 1059 are not relatively prime, so neither has an inverse modulo the other. In particular, they are both divisible by 353:

\[
gcd(1059, 1412) = gcd(1059, 1412 - 1059) = gcd(1059, 353) = 353.
\]
Problem 3 (Scheduling) (15 points).
The following DAG describes the prerequisites among tasks \{1, \ldots, 9\}.

![DAG Diagram]

(a) If each task takes unit time to complete, what is the minimum parallel time to complete all the tasks? Briefly explain.

Solution. 4. This is the size of a maximum chain, for example, 1238.

(b) What is the minimum parallel time if no more than two tasks can be completed in parallel? Briefly explain.

Solution. 5. There are 9 tasks and two processors, so time at least \(\lceil 9/2 \rceil = 5\) is required. A schedule that achieves this is 14, 26, 57, 39, 8.

Problem 4 (Probable Satisfiability) (15 points).
Truth values for propositional variables \(P, Q, R\) are chosen independently, with

\[
\Pr[P = T] = 1/2, \quad \Pr[Q = T] = 1/3, \quad \Pr[R = T] = 1/5.
\]

What is the probability that the formula

\[
(P \implies Q) \implies R
\]

is true?
Consider the separate cases in which (1) is true:

\( R = T \): This case happens with probability \( \frac{1}{5} \).

\( R = F \) AND \( Q = F \) AND \( P = T \): This case happens with probability \( \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{7}{15} \).

Another breakdown into cases is \( P = T \) AND \( Q = F \), and \( \text{NOT}(P = T \text{ AND } Q = F) \) AND \( R = T \) with probability

\[
\frac{1}{2} \cdot \frac{2}{3} + \left(1 - \frac{1}{2} \cdot \frac{2}{3} \right) \frac{1}{5} = \frac{7}{15}
\]

**STAFF NOTE:** (a) 8pts, (b) 12pts

**Problem 5 (Variance) (20 points).**

You are playing a game where you get \( n \) turns. Each of your turns involves flipping a coin a number of times. On the first turn, you have 1 flip, on the second turn you have two flips, and so on until your \( n \)th turn when you flip the coin \( n \) times. All the flips are mutually independent.

The coin you are using is biased to flip Heads with probability \( p \). You win a turn if you flip all Heads. Let \( W \) be the number of winning turns.

(a) Write a closed-form (no summations) expression for \( \text{Ex}[W] \).

**Solution.**

\[
\frac{1 - p^{n+1}}{1 - p} - 1.
\]

This can also be expressed as

\[
\frac{p(1 - p^n)}{1 - p}.
\]

Let \( H_k \) be indicator for winning the \( k \)th try. This means that

\[
W = \sum_{k=1}^{n} H_k.
\]

By independence of coin flips,

\[
p^k = \text{Pr}[H_k = 1] = \text{Ex}[H_k].
\]

so

\[
\text{Ex}[W] = \sum_{k=1}^{n} \text{Ex}[H_k] = \sum_{k=1}^{n} p^k = \left(\sum_{k=0}^{n} p^k\right) - 1 = \frac{1 - p^{n+1}}{1 - p} - 1.
\]

(b) Write a closed-form expression for \( \text{Var}[W] \).
Solution.

\[
\frac{1 - p^{n+1}}{1 - p} - \frac{1 - p^{2(n+1)}}{1 - p^2},
\]

which can also be written

\[
p \frac{1 - p^n}{1 - p} - p^2 \frac{1 - p^{2n}}{1 - p^2}.
\]

We have \( \text{Var}[H_k] = p^k(1 - p^k) \). Variances add because of mutual independence, so

\[
\text{Var}[W] = \sum_{k=1}^{n} p^k (1 - p^k)
\]

\[
= \sum_{k=1}^{n} p^k - p^{2k}
\]

\[
= \sum_{k=0}^{n} p^k - p^{2k}
\]

\[
= \sum_{0}^{n} p^k - \sum_{0}^{n} p^{2k}
\]

\[
= \frac{1 - p^{n+1}}{1 - p} - \frac{1 - p^{2(n+1)}}{1 - p^2}
\]

\[
\square
\]

Problem 6 (State Machines, Simple Graphs) (20 points).

Starting with some simple graph \( G \) with two or more vertices, keep applying the following operations: pick two vertices \( u \neq v \) such that either

(i) there is an edge between \( u \) and \( v \), and there is also a path from \( u \) to \( v \) which does not include this edge; in this case, delete the edge \( (u-v) \).

(ii) there is no path from \( u \) to \( v \); in this case, add the edge \( (u-v) \).

Continue until it is no longer possible to find two vertices \( u \neq v \) to which an operation applies.

(a) Let \( c \) be the number of connected components and \( e \) the number of edges of a simple graph. Prove that

\[
2c + e
\]

is a strictly decreasing derived variable for this process.

Solution. Operation (i) decreases \( e \) by one and leaves \( c \) unchanged since \( u \) and \( v \) remain connected. So \( 2c + e \) decreases by one.

Operation (ii) decreases \( c \) by one since the two connected components it connects become a single component. It increases \( e \) by one. So \( 2c + e \) changes by \((2 \cdot -1) + 1 = -1\), that is, it decreases by one.

(b) Explain why, starting from any finite simple graph, the procedure above terminates.
Solution. Since each step decreases $2c + e$ by one, the largest possible number of steps before termination is the initial value of $2c + e$.

More abstractly, $2c + e$ is a nonnegative integer-valued, strictly decreasing derived variable, so Theorem 6.3.2 implies termination.

(c) Explain why the procedure terminates with a tree.

Solution. We use the characterization of a tree as a cycle-free, connected, simple graph.

A final graph must be connected, because otherwise there would be two vertices with no path between them, and then a transition adding the edge between them would be possible.

A final graph can’t have a cycle, because deleting any edge on the cycle would be a possible operation.

Problem 7 (Infinite Bijection) (20 points).

The set $\{1, 2, 3\}^\omega$ consists of the infinite sequences of the digits 1, 2, and 3, and likewise $\{4, 5\}^\omega$ is the set of infinite sequences of the digits 4, 5. For example

\[
\begin{align*}
123123123\ldots & \in \{1, 2, 3\}^\omega, \\
2222222222\ldots & \in \{1, 2, 3\}^\omega, \\
45544554444\ldots & \in \{4, 5\}^\omega.
\end{align*}
\]

(a) Give an example of a total injective function

\[
f : \{1, 2, 3\}^\omega \rightarrow \{4, 5\}^\omega.
\]

Solution. Code each of the digits 1, 2, 3 into a pair of digits 4, 5, for example

\[
\begin{align*}
1 & \leftrightarrow 44 \\
2 & \leftrightarrow 45 \\
3 & \leftrightarrow 55,
\end{align*}
\]

and apply this digit by digit to map a sequence in $\{1, 2, 3\}^\omega$ to one in $\{4, 5\}^\omega$. For example.

\[
f(1231133\ldots) = 444554445555\ldots.
\]

(b) Give an example of a bijection $g : (\{1, 2, 3\}^\omega \times \{1, 2, 3\}^\omega) \rightarrow \{1, 2, 3\}^\omega$.

Solution. Let $g$ be the interleaving operation on two sequences, that is

\[
g(x_0x_1x_2x_3\ldots, y_0y_1y_2y_3\ldots) := x_0y_0x_1y_1x_2y_2x_3\ldots.
\]

(c) Explain why there is a bijection between $\{1, 2, 3\}^\omega \times \{1, 2, 3\}^\omega$ and $\{4, 5\}^\omega$. (You need not explicitly define the bijection.)
Solution. The composition $f \circ g$ is a total injective function, so

$$(\{1, 2, 3\}^\omega \times \{1, 2, 3\}^\omega) \inj \{4, 5\}^\omega.$$  

Conversely, mapping a sequence $s \in \{4, 5\}^\omega$ to the sequence $h(s)$ obtained by replacing 4’s by 1’s and 5’s by 2’s defines a total injective function

$$h : \{4, 5\}^\omega \to \{1, 2, 3\}^\omega.$$  

Now

$g^{-1} \circ h : \{4, 5\}^\omega \to (\{1, 2, 3\}^\omega \times \{1, 2, 3\}^\omega)$

is a total injective function, so

$$\{4, 5\}^\omega \inj (\{1, 2, 3\}^\omega \times \{1, 2, 3\}^\omega)$$

The Schröder-Bernstein Theorem 8.1.4 now implies that

$$\{4, 5\}^\omega \bij (\{1, 2, 3\}^\omega \times \{1, 2, 3\}^\omega)$$

Staff Note: (a) 8pts, 2pts each for the Not’s and 1pt each for the V’s; (b) 5 pts, 2pts for Not, 1pt for the V’s; (c) 6pts, 2 each item (d),(e) 3 pts each, -1pt for one mistake, -3pts for two or more mistakes

Problem 8 (Number Theory, Graphs) (25 points).

For each of the statements below in parts (a) through (c), write “V” for Valid or “N” for Not Valid, and provide counterexamples for those that are Not Valid. All variables $a, b, c, m, n$ range over positive integers.

(a) The following statements about the greatest common divisor:

- If $\gcd(a, b) \neq 1$ and $\gcd(b, c) \neq 1$, then $\gcd(a, c) \neq 1$.

  Solution. Not, $a = 2, b = 2 \cdot 3, c = 3$  

- If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

  Solution. Valid.

- $\gcd(a^n, b^n) = (\gcd(a, b))^n$.

  Solution. Valid.

- $\gcd(ab, ac) = a \gcd(b, c)$.

  Solution. Valid.

- If some integer linear combination of $a$ and $b$ equals 1, then so does some integer linear combination of $a$ and $b^2$. 

  Solution. Valid.
Solution. Valid.

- If some integer linear combination of \(a\) and \(b\) equals 2, then so does some integer linear combination of \(a\) and \(b^2\).

  Solution. Not, let \(a = 4, b = 2\).

(b) The following statements about congruence modulo \(n\), where \(n > 1\):

- If \(ac \equiv bc \mod n\) and \(n\) does not divide \(c\), then \(a \equiv b \mod n\).

  Solution. Not. Need \(c\) relatively prime to \(n\). Counterexample: \(n = 2 \cdot 3, a = 0, b = 2, c = 3\)

- If \(a \equiv b \mod nm\), then \(a \equiv b \mod n\), for \(m, n > 1\).

  Solution. Valid.

- For relatively prime \(m, n > 1\),
  \[
  [a \equiv b \mod m] \text{ AND } a \equiv b \mod n] \iff [a \equiv b \mod mn]
  \]

  Solution. Valid.

- If \(a, b > 1\), then
  \[
  [a \text{ has a multiplicative inverse } \mod b \iff b \text{ has a multiplicative inverse } \mod a].
  \]

  Solution. Valid. \(a\) has a multiplicative inverse \(\mod b\) iff \(a, b\) relatively prime iff \(b\) has a multiplicative inverse \(\mod a\).

(e) The following statements about a finite simple graph \(G\):

- \(G\) has a spanning tree.

  Solution. Not. Any disconnected graph is a counterexample.

- \(|V(G)| = O(|E(G)|)\) for connected \(G\).

  Solution. Valid. To be connected, there must be at least \(|V(G)| - 1\) edges.

- \(|E(G)| = O(|V(G)|)\).

  Solution. Not. \(|V(K_n)| = n = o(n^2)\), but \(|E(K_n)| = \Theta(n^2)\).

(d) Circle all the properties below that are preserved under graph isomorphism:

- The vertices can be numbered 1 through 7.
- There is a cut edge.
- Two edges are of equal length.
- The XOR of two properties that are preserved under isomorphism.

  Solution. All are preserved except “Two edges are of equal length.”

(e) A sink in a digraph is a vertex whose only edge is a self-loop. Circle all of the assertions below that are true of stationary distributions of random walk graphs with exactly two sinks:
- There is 2-sink graph with no stationary distribution.
- There is a 2-sink graph with a unique stationary distribution.
- There is a 2-sink graph with an uncountable number stationary distributions.
- Every 2-sink graph has an uncountable number of stationary distributions.

Solution. The first two choices are false, and the last two are true. That’s because a distribution in which one sink has probability \( r \in [0, 1] \subseteq \mathbb{R} \) and the other sink has probability \( 1 - r \) is stable, and there is an uncountable number of real numbers in \([0, 1]\).

STAFF NOTE: (a),(b): 4 pts each; (c),(d) 6pts each

Problem 9 (Counting) (20 points).
We want to count the number of length-\( n \) binary strings in which the substring 011 occurs in various places. For example, the length-14 string

\[
0010011001101,
\]

has 011 in the 4th position and the 8th position. (Note that by convention, a length-\( n \) string starts with position zero and ends with position \( n - 1 \).) Assume \( n \geq 7 \).

(a) Let \( r \) be the number of length-\( n \) binary strings in which 011 occurs starting at the 4th position. Write a formula for \( r \) in terms of \( n \).

Solution.

\[
r = 2^{n-3}.
\]

This is the number of patterns of the remaining \( n - 3 \) bits besides the substring 011 occupying positions 4–6.

(b) Let \( A_i \) be the set of length-\( n \) binary strings in which 011 occurs starting at the \( i \)th position. (\( A_i \) is empty for \( i > n - 3 \).) If \( i \neq j \), the intersection \( A_i \cap A_j \) is either empty or of size \( s \). Write a formula for \( s \) in terms of \( n \).

Solution.

\[
s = 2^{n-6}.
\]

To be nonempty, the copies of 011 at \( i \) and \( j \) use up 6 positions, leaving \( n - 6 \) positions that can contain any pattern of bits. So \(|A_i \cap A_j| = 2^{n-6}|.

(c) Let \( t \) be the number of pairs \((i, j)\) such that \( A_i \cap A_j \) is nonempty, where \( 0 \leq i < j \). Write a binomial coefficient for \( t \) in terms of \( n \).

Solution.

\[
t = \binom{n-4}{2}.
\]

This is the same as asking how many ways there are to place two copies of 011 in a length \( n \) binary sequence. Since the copies can’t overlap, this is the same as the number of sequences with \( n - 6 \) 1’s and two
0’s, where the 1’s indicate positions not occupied by the two copies and the 0’s indicate where the copies are placed. By the Bookkeeper Principle, this is
\[ \binom{(n - 6) + 2}{2}. \]

(d) How many length 9 binary strings are there that contain the substring 011? You should express your answer as an integer or as a simple expression which may include the above constants, \( r, s \) and \( t \) for \( n = 9 \).

Hint: Inclusion-exclusion for \( \bigcup_0^8 A_i \).

Solution.
\[
\left| \bigcup_0^8 A_i \right| = 7 \cdot r - t \cdot s + 1 = 369. \tag{2}
\]

By Inclusion-exclusion
\[
\left| \bigcup_0^8 A_i \right| = \sum_0^8 |A_i| - \sum_{i \neq j} |A_i \cap A_j| + \sum_{i \neq j \neq k} |A_i \cap A_j \cap A_k|. \tag{3}
\]

Since \( A_7 = A_8 = \emptyset \), there are 7 terms in the first sum in (3), and each term is \( r \).

There are \( t \) terms in the second sum in (3), each of size \( s \).

Finally, among the terms in the third sum,
\[
A_0 \cap A_3 \cap A_6 = \{011011011\},
\]
and all the other intersections are empty, so the third term is 1. This leads to equation (2).

STAFF NOTE: -5 for using 1/2 instead of p/q.

Problem 10 (Expectation) (15 points).
A coin with probability \( p \) of flipping Heads and probability \( q := 1 - p \) of flipping tails is repeatedly flipped until three consecutive Heads occur. The outcome tree, \( D \), for this setup is illustrated in Figure 1.

Let \( e(S) \) be the expected number of flips starting at the root of subtree \( S \) of \( D \). So we’re interested in finding \( e(D) \).

Write a small system of equations involving \( e(D) \), \( e(B) \), and \( e(C) \) that could be solved to find \( e(D) \). You do not need to solve the equations.

Solution. By the Total Expectation Rule, we have
\[
e(D) = q(1 + e(D)) + p(1 + e(B)) = 1 + q \cdot e(D) + p \cdot e(B),
\]
\[
e(B) = q(1 + e(D)) + p(1 + e(C)) = 1 + q \cdot e(D) + p \cdot e(C),
\]
\[
e(C) = q(1 + e(D)) + p(1 + 0) = 1 + q \cdot e(D).
\]
A solution to these equations was not called for, but is easy to work out. Namely, substituting for $e(C)$, we get

$$e(B) = 1 + q e(D) + p(1 + q e(D)) = 1 + p + q e(D)(1 + p)$$

and then substituting this expression for $e(B)$, we get

$$e(D) = 1 + q e(D) + p(1 + p + q e(D)(1 + p))$$
$$= 1 + p + p^2 + q e(D)(1 + p + p^2)$$
$$= (1 + p + p^2)(1 + q e(D))$$
$$= \frac{1 - p^3}{q}(1 + q e(D))$$
$$= (1 - p^3) \left( \frac{1}{q} + e(D) \right)$$

so

$$e(D)p^3 = \frac{1 - p^3}{q}$$

and

$$e(D) = \frac{1 - p^3}{qp^3}.$$

For $p = 1/2$, this would be 14.

STAFF NOTE: (a) 3pts; (b) 4 pts; (c),(d)(e) 6pts each

Problem 11 (Chebyshev Bound) (25 points).
You have a biased coin which flips Heads with probability $p$. You flip the coin $n$ times. The coin flips are all mutually independent. Let $H$ be the number of Heads.

(a) Write a simple expression in terms of $p$ and $n$ for $\text{Ex}[H]$, the expected number of Heads.
Solution. \( np \).

**STAFF NOTE:** Not needed for full credit:

Let \( H_i \) be the indicator variable that is 1 if and only if the \( i \)th coin flip comes out Heads (and 0 otherwise). Then

\[
H = H_1 + H_2 + \cdots + H_n.
\]

Hence, by linearity of expectation,

\[
\text{Ex}[H] = \text{Ex}[H_1 + H_2 + \cdots + H_n] = \text{Ex}[H_1] + \cdots + \text{Ex}[H_n].
\]

The expectation of an indicator variable is the probability it equals 1. Hence, \( \text{Ex}[H_i] = p \). We conclude that \( \text{Ex}[H] = n \cdot p \).

(b) Write a simple expression in terms of \( p \) and \( n \) for \( \text{Var}[H] \), the variance of the number of Heads.

Solution. \( np(1 - p) \).

**STAFF NOTE:** Not needed for full credit:

By the independence of the \( H_i \), we know

\[
\text{Var}[H] = \text{Var}[H_1 + \cdots + H_n] = \text{Var}[H_1] + \cdots + \text{Var}[H_n].
\]

Finally, we know the variance of an indicator with expectation \( p \) is \( p(1 - p) \).

(c) Write a simple expression in terms of \( p \) for the upper bound that Markov’s Theorem gives for the probability that the number of Heads is larger than the expected number by at least 1\% of the number of flips, that is, by \( n/100 \).

Solution.

\[
\frac{100p}{100p + 1}.
\]

We want the Markov bound on

\[
\Pr[H \geq \text{Ex}[H] + \frac{n}{100}].
\]

The Markov bound on the probability of being at least \( a\mu \) is \( 1/a \). So we want

\[
a\mu = \mu + \frac{n}{100}.
\]

Solving for \( a \) gives

\[
a = 1 + \frac{n}{100\mu} = 1 + \frac{n}{100pn} = 1 + \frac{1}{100p} = \frac{100p + 1}{100p}.
\]
(d) Show that the bound Chebyshev’s Theorem gives for the probability that $H$ differs from $\text{Ex}[H]$ by at least $n/100$ is

$$100^2 \frac{p(1-p)}{n}.$$ 

Solution.

$$\Pr \left[ \left| H - \text{Ex}[H] \right| \geq \frac{n}{100} \right] \leq \frac{\text{Var}[H]}{(n/100)^2} \quad \text{(by Chebyshev’s Theorem)}$$

$$= \frac{np(1-p)}{(n/100)^2}.$$ 

Simplifying the right hand expression gives the stated bound. 

(e) The bound in part (d) implies that if you flip at least $m$ times for a certain number $m$, then there is a 95% chance that the proportion of Heads among these $m$ flips will be within 0.01 of $p$. Write a simple expression for $m$ in terms of $p$.

Solution.

$$20 \cdot 100^2 p(1-p).$$

The average number Heads is within 0.01 of $p$ iff the total number, $n$, of flips is with $n/100$ of the expectation $pn$. So letting $u :=$ the answer to part (d). We need

$$u \leq \frac{1}{20}.$$ 

Solving for $n$, we get the answer. 

Problem 12 (Graphs, Logic, Probability) (20 points).

Let $G$ be a simple graph with $n$ vertices. Let “$A(u, v)$” mean that vertices $u$ and $v$ are adjacent, and let “$W(u, v)$” mean that there is a length-two walk between $u$ and $v$.

(a) Explain why $W(u, u)$ holds iff $\exists v. A(u, v)$.

Solution. A length two path from $u$ to $u$ must be a path that goes back and forth on some edge incident to $u$. 

(b) Write a predicate-logic formula defining $W(u, v)$ in terms of the predicate $A(\ldots)$ when $u \neq v$.

Solution.

$$W(u, v) ::= \exists t. A(u, t) \text{ AND } A(t, v).$$

This formula actually works even if $u = v$. 

There are $e ::= \binom{n}{2}$ possible edges between the $n$ vertices of $G$. Suppose the actual edges of $E(G)$ are chosen randomly from this set of $e$ possible edges. Each edge is chosen with probability $p$, and the choices are mutually independent.

(c) Write a simple formula in terms of $p, e, \text{ and } k$ for $\Pr[|E(G)| = k]$. 
The number of chosen edges has a binomial distribution.

\[(d)\] Write a simple formula in terms of \(p\) and \(n\) for \(\Pr[W(u, u)]\).

**Solution.**

\[
\left(\frac{e}{k}\right) p^k (1 - p)^{e-k}.
\]

Let \(w, x, y\) and \(z\) be four distinct vertices.

Because edges are chosen mutually independently, events that depend on disjoint sets of edges will be mutually independent. For example, the events

\[A(w, y) \text{ AND } A(y, x)\]

and

\[A(w, z) \text{ AND } A(z, x)\]

are independent since \(\{w\to y\}, \{y\to x\}, \{w\to z\}, \{z\to x\}\) are four distinct edges.

\[(e)\] Let

\[r := \Pr[\text{NOT}(W(w, x))],\]

where \(w\) and \(x\) are distinct vertices. Write a simple formula for \(r\) in terms of \(n\) and \(p\).

**Hint:** Different length-two paths between \(x\) and \(y\) don’t share any edges.

**Solution.**

\[r = (1 - p^2)^{n-2}.
\]

Let \(Z := V(G) - \{w, x\}\) be the set of \(n - 2\) vertices other than \(w\) and \(x\).

\[
\Pr[\text{NOT}(W(w, x))] = \Pr[\forall z \in Z, \text{ NOT}(A(w, z) \text{ AND } A(z, x))] \\
= \prod_{z \in Z} \Pr[\text{NOT}(A(w, z) \text{ AND } A(z, x))] \\
= \prod_{z \in Z} (1 - \Pr[A(w, z) \text{ AND } A(z, x)]) \\
= \prod_{z \in Z} (1 - \Pr[A(w, z)] \cdot \Pr[A(z, x)]) \\
= \prod_{z \in Z} (1 - p^2) \\
= (1 - p^2)^{n-2}.
\]
Problem 13 (Induction) (30 points).

The 2-3-averaged numbers are a subset, N23, of the real interval [0, 1] defined recursively as follows:

Base cases: 0, 1 ∈ N23.

Constructor case: If a, b are in N23, then so is L(a, b) where
\[ L(a, b) := \frac{2a + 3b}{5}. \]

(a) Use ordinary induction or the Well-Ordering Principle to prove that
\[ \left( \frac{3}{5} \right)^n \in N23 \]
for all nonnegative integers n. (Do not overlook part (b) on the next page.)

Solution. By induction on n with induction hypothesis as stated.

Base cases (n = 0): \((3/5)^0 = 1 \in N23\) by the base case of the recursive definition of N23.

Inductive step: We may assume by induction that \((3/5)^n \in N23\). Also 0 ∈ N23 by the base case of the recursive definition of N23. Therefore, \(L(0, (3/5)^n) = (3/5)^{n+1} \in N23\) by the Constructor case.

(b) Prove by Structural Induction that the product of two 2-3-averaged numbers is also a 2-3-averaged number.

Hint: Prove by structural induction on c that, if d ∈ N23, then cd ∈ N23.

Solution. The induction hypothesis is
\[ P(c) := \forall d \in N23. \ c \cdot d \in N23. \]

Base cases: \(P(0)\) holds since \(0 \cdot d = 0 \in N23\). Likewise, \(P(1)\) holds since \(1 \cdot d = d \in N23\).

Constructor case: \((c = L(a, b)\ for \ a, b \in N23)\). So
\[ c \cdot d = \frac{2a + 3b}{5} \cdot d = \frac{2(ad) + 3(bd)}{5} = L(ad, bd). \]

Now \(ad, bd \in N23\) by structural induction hypothesis, and therefore \(L(ad, bd) \in N23\) by the Constructor step, proving that \(c \cdot d \in N23\) as required.