Staff Solutions to In-Class Problems Week 5, Mon.

STAFF NOTE: Stable Marriage

Problem 1.

Four Students want separate assignments to four VI-A Companies. Here are their preference rankings:

<table>
<thead>
<tr>
<th>Student</th>
<th>Companies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Albert</td>
<td>HP, Bellcore, AT&amp;T, Draper</td>
</tr>
<tr>
<td>Sarah</td>
<td>AT&amp;T, Bellcore, Draper, HP</td>
</tr>
<tr>
<td>Tasha</td>
<td>HP, Draper, AT&amp;T, Bellcore</td>
</tr>
<tr>
<td>Elizabeth</td>
<td>Draper, AT&amp;T, Bellcore, HP</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Company</th>
<th>Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>AT&amp;T</td>
<td>Elizabeth, Albert, Tasha, Sarah</td>
</tr>
<tr>
<td>Bellcore</td>
<td>Tasha, Sarah, Albert, Elizabeth</td>
</tr>
<tr>
<td>HP</td>
<td>Elizabeth, Tasha, Albert, Sarah</td>
</tr>
<tr>
<td>Draper</td>
<td>Sarah, Elizabeth, Tasha, Albert</td>
</tr>
</tbody>
</table>

(a) Use the Mating Ritual to find two stable assignments of Students to Companies.

Solution. Treat Students as Boys and the result is the following assignment:

<table>
<thead>
<tr>
<th>Student</th>
<th>Companies</th>
<th>Rank in the original list</th>
</tr>
</thead>
<tbody>
<tr>
<td>Albert</td>
<td>Bellcore</td>
<td>2</td>
</tr>
<tr>
<td>Sarah</td>
<td>AT&amp;T</td>
<td>1</td>
</tr>
<tr>
<td>Tasha</td>
<td>HP</td>
<td>1</td>
</tr>
<tr>
<td>Elizabeth</td>
<td>Draper</td>
<td>1</td>
</tr>
</tbody>
</table>

Treat Companies as Boys and the result is the following assignment:

<table>
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<tr>
<th>Company</th>
<th>Students</th>
<th>Rank in the original list</th>
</tr>
</thead>
<tbody>
<tr>
<td>AT&amp;T</td>
<td>Albert</td>
<td>2</td>
</tr>
<tr>
<td>Bellcore</td>
<td>Sarah</td>
<td>2</td>
</tr>
<tr>
<td>HP</td>
<td>Tasha</td>
<td>2</td>
</tr>
<tr>
<td>Draper</td>
<td>Elizabeth</td>
<td>2</td>
</tr>
</tbody>
</table>

(b) Describe a simple procedure to determine whether any given stable marriage problem has a unique solution, that is, only one possible stable matching. Briefly explain why it works.
Solution. See if the Mating Ritual with Boys as suitors yields the same solution as the algorithm with Girls as suitors. These two marriage assignments are respectively boy-optimal and boy-pessimal (Theorem 6.4.10).

To see why this implies uniqueness, suppose Alice is both Harry’s optimal wife and also his pessimal wife. Now in any set of stable marriages, if Harry had a wife other than Alice, that wife would either be better than Alice or worse than Alice, which is impossible. So Alice is Harry’s only feasible wife. So when optimal and pessimal marriages are the same, every boy has only one feasible wife, which means only one set of stable marriages is possible.

There is also a simple alternative argument\(^1\) that does not depend on the fact that Boy-optimal is Girl-pessimal. Namely, suppose a couple is married in both the Girl-optimal and Boy-optimal stable marriage sets. Now if they were not married in some other stable marriage set, then since each is optimal for the other, they would prefer each other to their spouses; that is, they would be a rogue couple, contradicting stability of this other marriage set.

These arguments actually yield a stronger result than stated: if a boy is married to the same girl in both the Boy-optimal and Girl-optimal marriage sets, then this girl is his only feasible wife. This holds even if the Boy-optimal and Girl-optimal matchings are not completely the same.

**STAFF NOTE**: Students usually can figure out to look for the same solution when Boys’ and Girls’ roles are reversed, but they often come up with vague and unconvincing arguments for uniqueness like “If a set of marriages is optimal for boys and also for girls, it must be unique.” This leaves unexplained why it is unique—for example, why is it impossible to have another stable set that was suboptimal for Boys and for Girls?

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**Problem 2.**

The Mating Ritual for finding stable marriages works even when the numbers of men and women are not equal. As before, a set of (monogamous) marriages between men and women is called stable when it has no “rogue couples.”

(a) Extend the definition of *rogue couple* so it covers the case of unmarried men and women. Verify that in a stable set of marriages, either all the men are married or all the women are married.

**Solution.** A rogue couple for a given a set of marriages is a woman, Alice, and a man, Bob, such that

- Alice is unmarried or has a spouse she likes less than Bob, and also
- Bob is unmarried or has a spouse he likes less than Alice.

By this definition, if both a man and a woman are unmarried, they are a rogue couple, so in a stable set of marriages, either all the men are married or all the women are married.

(b) Explain why even in the case of unequal numbers of men and women, applying the Mating Ritual will yield a stable matching.

**Solution.** The preserved invariant\(^2\) and termination proof for the Mating Ritual apply in this case without change, so we just need to verify that there are no rogue couples of the more general kind in part (a).

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\(^1\)Suggested by Sravya Bhamidipati, September, 2015.

\(^2\)For every woman, \(w\), and every man, \(m\), if \(w\) is crossed off \(m\)’s list, then \(w\) has a suitor whom she prefers over \(m\).
As before, on the Wedding Day, no man will be in a rogue couple with a woman not on his final list because, by the preserved invariant, any such woman has a spouse she prefers to him. But no man will be in a rogue couple with a woman still on his final list because he must be married to that woman or to a woman higher on his list who, by definition, he finds more preferable. (We know that the man in this last case must actually be married, because if he weren’t, he would be crossing off the top-ranked woman on his list and continuing to serenade his next choice tomorrow, and the Ritual would not have ended yet.)

Problem 3.
The most famous application of stable matching was in assigning graduating medical students to hospital residencies. Each hospital has a preference ranking of students, and each student has a preference ranking of hospitals, but unlike finding stable marriages between an equal number of boys and girls, hospitals generally have differing numbers of available residencies, and the total number of residencies may not equal the number of graduating students.

Explain how to adapt the Stable Matching problem with an equal number of boys and girls to this more general situation. In particular, modify the definition of stable matching so it applies in this situation, and explain how to adapt the Mating Ritual to handle it.

Solution. The idea is to treat the residencies rather than the hospitals as men (or women) in the Mating Ritual. So a “matching” will be an assignment of students to residencies. More precisely, a matching is an injection, $A : \text{students} \to \text{residencies}$.

An assignment is “stable” when it has no rogue couples, where a rogue couple is defined with the idea that a hospital would rather have a residency filled than unfilled—even if they gave lowest rank to the student who would be assigned to the residency, and a student would rather have some residency than none—even if the residency was at their lowest ranked hospital.

More precisely, for any residency, $R$, let $H(R)$ be the hospital where $R$ is located. Then a rogue couple for an assignment $A$ is a residency/student pair $(R, S)$ such that

- $S$ has no residency assignment ($S \notin \text{domain}(A)$), or $S$ likes $H(R)$ better than his assigned hospital $H(A(S))$, and

- $R$ has no assigned student ($R \notin \text{range}(A)$), or $A(S') = R$ for some student $S'$ that $H(R)$ ranks lower than $S$.

Notice that by this definition, $(R, S)$ will be a rogue couple when both $R$ has no assigned student and $S$ is not assigned to a residency. So in a stable assignment, either all residencies have students assigned to them ($A$ is a surjection), or all students have assigned residencies ($A$ is total).

Now the Mating Ritual can be carried out with students playing the role of men and residencies as women (or vice versa). A stable assignment is bound to be the result, even if the number of students and number of residencies are not the same (see Problem 6.21).

Problem 4.
Give an example of a stable matching between 3 boys and 3 girls where no person gets their first choice. Briefly explain why your matching is stable. Can your matching be obtained from the Mating Ritual or the Ritual with boys and girls reversed?

Solution. The idea is to let all the boys have different first choices and all the girls have different first choices, and each person’s first choice ranks them last. Moreover, let all the boys also have different second choices.
Now having each boy marry his second choice will be stable: a boy, Tom, can only be in a rogue couple with his first choice girl, Nicole. But since Tom is Nicole’s last choice, she will be married to someone she prefers to Tom, and so won’t have a rogue relationship with Tom.

For example with boys 1, 2, 3 and the girls a, b, c, the preferences could be:

<table>
<thead>
<tr>
<th>choice</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>2</td>
<td>b</td>
<td>c</td>
<td>a</td>
</tr>
<tr>
<td>3</td>
<td>c</td>
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<td>c</td>
<td>a</td>
<td>b</td>
</tr>
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</table>

So the second-choice matching (1, b), (2, c), (3, a) will be stable even though no person gets their first choice. Moreover, this matching won’t be one that comes from the Mating Ritual since it neither boy optimal nor girl optimal: giving boys their first choice would obviously be boy optimal, and giving girls their first choice would be girl optimal.