Staff Solutions to In-Class Problems Week 12, Fri.

STAFF NOTE: Mutual Independence, Confidence, Ch.18.8-18.9

Problem 1.
Suppose you flip three fair, mutually independent coins. Define the following events:

- Let $A$ be the event that the first coin is heads.
- Let $B$ be the event that the second coin is heads.
- Let $C$ be the event that the third coin is heads.
- Let $D$ be the event that an even number of coins are heads.

(a) Use the four step method to determine the probability space for this experiment and the probability of each of $A$, $B$, $C$, $D$.

Solution. The tree is a binary tree with depth 3 and 8 leaves. The successive levels branch to show whether or not the successive events $A$, $B$, $C$ occur. By the definitions of the characteristics fair and independent, each branch from a vertex is equally likely to be followed. So the probability space has, as outcomes, eight length-3 strings of H’s and T’s, each of which has probability $(1/2)^3 = 1/8$.

Each of the events $A$, $B$, $C$, $D$ are true in four of the outcomes and hence has probability 1/2.

(b) Show that these events are not mutually independent.

Solution.

$$\Pr[A \cap B \cap C \cap D] \neq (1/2)^4 = \Pr[A] \cdot \Pr[B] \cdot \Pr[C] \cdot \Pr[D].$$

(c) Show that they are 3-way independent.

Solution. 3-way independence requires that any subset of three events must be mutually independent. Because the coin tosses are mutually independent, we know that $A$, $B$, $C$ are mutually independent. So we must also check that any three events including $D$ are mutually independent. By symmetry, we only need to check whether the three events $A$, $B$, $D$ are mutually independent. First, let’s check

$$\Pr[A \cap B \cap D] = \Pr[A] \cdot \Pr[B] \cdot \Pr[D],$$

which follows because

$$\Pr[A \cap B \cap D] = \Pr[\{HHT\}] = \frac{1}{8} = \Pr[A] \cdot \Pr[B] \cdot \Pr[D].$$
We must also check that $A, B, D$ are 2-way (pairwise) independent. We are given that $A$ and $B$ are independent. $A$ and $D$ are independent because
\[
\Pr[A \cap D] = \Pr[HHT, HTH] = \frac{1}{4} = \Pr[A] \cdot \Pr[D].
\]
We conclude by symmetry that $B$ and $D$ are also independent.
The above completes the verification that $A, B, C, D$ are 3-way independent.

Problem 2.
A somewhat reliable allergy test has the following properties:

- If you are allergic, there is a 10% chance that the test will say you are not.
- If you are not allergic, there is a 5% chance that the test will say you are.

(a) The test results are correct at what confidence level?

Solution. The confidence is the smaller of the probability of a correct allergic diagnosis and a correct non-allergic diagnosis, namely,
\[
\min(1 - 1/10, 1 - 1/20) = 9/10.
\]
So the test is correct at the 90% confidence level.

(b) What is the Bayes factor for being allergic when the test diagnoses a person as allergic?

Solution. IF $H$ is the event of being allergic, and $E$ is the event of being diagnosed as allergic, then
\[
\text{Bayes-factor}(E, H) := \frac{\Pr[E | H]}{\Pr[E | \overline{H}]} = \frac{9/10}{1/20} = 18.
\]

(c) What can you conclude about the odds of a random person being allergic given that the test diagnoses them as allergic?

Solution. Your odds of being allergic are 18 times those of the general population. But without knowing the fraction of allergic people in the population, we can’t tell what would be the odds of being allergic given an allergic diagnosis.

(d) How would your doctor calculate these odds of being allergic based on what’s known about the allergy test?

Solution. The odds of a random person being allergic are one to three. The odds of being allergic given that someone tests as allergic are 18 times as large, namely, $18 \cdot 1/3 = 6$. So the odds are six to one that an allergic diagnosis is correct.
(e) Another doctor reviews your test results and medical record and says your odds of being allergic are really much higher, namely thirty-six to one. Briefly explain how two conscientious doctors could disagree so much. Is there a way you could determine your actual odds of being allergic?

Solution. The doctors could disagree because they are seeing you as a representative of two different groups. Your first doctor just views you as a member of the general population, so he reports your six to one odds based on the one to three odds that a random person is allergic.

Your second doctor might observe from your medical record that you had measles as a child and that 2/3 of people who had childhood measles are known to be allergic. So viewing you as a representative of the childhood measles population, your second doctor would recognize that the odds of allergy in this population are two to one. With the Bayes factor of 18, the odds of really being allergic for someone who tested as allergic and also had childhood measles becomes 18 times two to one, namely thirty-six to one.

This disagreement illustrates that there are no “true” odds that you are allergic. Either you are allergic or you are not; there are no odds about it. The statements about “your” odds are really about the odds for random people with characteristics like yours. Attending to different characteristics will lead to different odds. In the end, no one besides yourself is exactly like you, and the only true odds of your being allergic are either zero if you are not allergic or infinity if you are allergic.

Problem 3.

An International Journal of Pharmacological Testing has a policy of publishing drug trial results only if the conclusion holds at the 95% confidence level. The editors and reviewers always carefully check that any results they publish came from a drug trial that genuinely deserved this level of confidence. They are also careful to check that trials whose results they publish have been conducted independently of each other.

The editors of the Journal reason that under this policy, their readership can be confident that at most 5% of the published studies will be mistaken. Later, the editors are embarrassed—and astonished—to learn that every one of the 20 drug trial results they published during the year was wrong. The editors thought that because the trials were conducted independently, the probability of publishing 20 wrong results was negligible, namely, \((\frac{1}{20})^{20} < 10^{-25}\).

Write a brief explanation to these befuddled editors explaining what’s wrong with their reasoning and how it could be that all 20 published studies were wrong.

Hint: xkcd comic: “significant” xkcd.com/882/

Solution. An assertion of 95% confidence means that if very many trials were carried out, we expect that close to 95% of the trials would yield a correct conclusion. So if a random sample of drug test results were submitted for publication, then the editors would be correct in expecting that only 5% of them would be wrong.

But that’s not what happens: not all the trials are written up and submitted—only the interesting ones get submitted. In this context, 95% confidence is not very important—remember that a phoney weatherman can predict sunshine in the Sahara desert with much more than 95% confidence. What matters is the rate of false positives—when an ineffective drug is declared effective. We’ve seen examples—the TB test in Section 18.9—where the probability that a positive finding is correct can be much lower than the confidence level of the test.

For example, there may be more than 400 worthless “alternative” drugs being tested by proponents who are genuinely honest, if misguided. When they conduct careful trials correct at a 95% confidence level, we can expect that in twenty of the 400 trials, worthless—even damaging—drugs will falsely be declared effective. The remaining 380 of the 400 trials—which correctly identified their drug as ineffective—would not be submitted for publication because the trials did not find a drug worth attending to. But the twenty trials that mistakenly showed positive results might well all be submitted by honest researchers with no intention to mislead.
This is why, unless there is an explanation of why a therapy works, scientists and doctors usually doubt trials claiming to confirm the efficacy of some far-fetched medication at a high confidence level. It is also why some medical regulatory agencies are pushing for a new policy that results of all clinical trials be published, not just the ones that show positive results. This policy would validate the editors’ original expectation that only 5% of their published studies would be mistaken, but then 95% of the published papers would be uninteresting.

**Problem 4.**
Event $E$ is *evidence in favor* of event $H$ when $\Pr[H \mid E] > \Pr[H]$, and it is *evidence against* $H$ when $\Pr[H \mid E] < \Pr[H]$. 

(a) Give an example of events $A, B, H$ such that $A$ and $B$ are independent, both are evidence for $H$, but $A \cup B$ is evidence against $H$.

*Hint:* Let $\mathcal{S} = [1..8]

**STAFF NOTE:** There may be simpler examples.

**Solution.**

\[
\begin{align*}
H & := \{3, 4, 8\} \\
A & := \{1, 2, 3, 4\} \\
B & := \{3, 4, 5, 6\}.
\end{align*}
\]

(b) Prove $E$ is evidence in favor of $H$ iff $\overline{E}$ is evidence against $H$.

**Solution.** To prove the implication from left to right, suppose to the contrary that $E$ was evidence for $H$ and $\overline{E}$ was not evidence against $H$. Then

\[
\begin{align*}
\Pr[H] &= \Pr[H \mid E] \Pr[E] + \Pr[H \mid \overline{E}] \Pr[\overline{E}] & \text{(Total probability)} \\
&\geq \Pr[H \mid E] \Pr[E] + \Pr[H] \Pr[\overline{E}] & \text{($\overline{E}$ is not evidence against $H$)} \\
&> \Pr[H] \Pr[E] + \Pr[H] \Pr[\overline{E}] & \text{($E$ is evidence in favor of $H$)} \\
&= \Pr[H] (\Pr[E] + \Pr[\overline{E}]) \\
&= \Pr[H].
\end{align*}
\]

a contradiction.

The implication from right to left now follows by replacing $E$ by $\overline{E}$ in the left to right implication. 
