Problem Set 6

Due: April 15

Reading:

- Chapter 12. Simple Graphs omitting Section 10.7.
- Chapter 14. Asymptotics through Section 14.5.

Problem 1.
Scholars through the ages have identified twenty fundamental human virtues: honesty, generosity, loyalty, prudence, completing the weekly course reading-response, etc. At the beginning of the term, every student in Math for Computer Science possessed exactly eight of these virtues. Furthermore, every student was unique; that is, no two students possessed exactly the same set of virtues. The Math for Computer Science course staff must select one additional virtue to impart to each student by the end of the term. Prove that there is a way to select an additional virtue for each student so that every student is unique at the end of the term as well.

Suggestion: Use Hall’s theorem. Try various interpretations for the vertices on the left and right sides of your bipartite graph.

Problem 2.
Determine which among the four graphs pictured in Figure 1 are isomorphic. For each pair of isomorphic graphs, describe an isomorphism between them. For each pair of graphs that are not isomorphic, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, prove that it is indeed preserved under isomorphism (you only need prove one of them).

Problem 3.
This problem generalizes the result proved Theorem 12.6.3 that any graph with maximum degree at most \( w \) is \((w + 1)\)-colorable.

A simple graph, \( G \), is said to have width \( w \) iff its vertices can be arranged in a sequence such that each vertex is adjacent to at most \( w \) vertices that precede it in the sequence. If the degree of every vertex is at most \( w \), then the graph obviously has width at most \( w \)—just list the vertices in any order.

(a) Prove that every graph with width at most \( w \) is \((w + 1)\)-colorable.

(b) Describe a 2-colorable graph with minimum width \( n \).

(c) Prove that the average degree of a graph of width \( w \) is at most \( 2w \).

(d) Describe an example of a graph with 100 vertices, width 3, but average degree more than 5.
Problem 4.
Prove Corollary 12.9.12: If all edges in a finite weighted graph have distinct weights, then the graph has a unique MST.

*Hint:* Suppose $M$ and $N$ were different MST's of the same graph. Let $e$ be the smallest edge in one and not the other, say $e \in M \setminus N$, and observe that $N + e$ must have a cycle.

Problem 5.
Is a Harvard degree really worth more than an MIT degree? Let us say that a person with a Harvard degree starts with $40,000 and gets a $20,000 raise every year after graduation, whereas a person with an MIT degree starts with $30,000, but gets a 20% raise every year. Assume inflation is a fixed 8% every year. That is, $1.08 a year from now is worth $1.00 today.

(a) How much is a Harvard degree worth today if the holder will work for $n$ years following graduation?

(b) How much is an MIT degree worth in this case?

(c) If you plan to retire after twenty years, which degree would be worth more?

Problem 6.
Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

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\sum_{i=1}^{\infty} \frac{1}{(2i + 1)^2}
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