Problem Set 5

Due: April 1

Reading:

- Chapter 9. Number Theory through Section 9.9.
- Chapter 10. Digraphs, (Section 10.7 optional).

Problem 1.
Prove that the greatest common divisor of three integers \(a, b,\) and \(c\) is equal to their smallest positive linear combination; that is, the smallest positive value of \(sa + tb + uc\), where \(s, t,\) and \(u\) are integers.

*Hint:* Let \(m\) be the smallest positive linear combination of \(a, b,\) and \(c\). Prove that \(\gcd(a, b, c) \leq m\) and \(m \geq \gcd(a, b, c)\).

Problem 2.
Suppose that \(a \equiv b \pmod{n}\) and \(n > 0\). Prove or disprove the following assertions:

(a) \(a^c \equiv b^c \pmod{n}\) where \(c \geq 0\)

(b) \(c^a \equiv e^b \pmod{n}\) where \(a, b \geq 0\)

Problem 3. (a) What are the maximal and minimal elements, if any, of the power set \(\text{pow}\{1, \ldots, n\}\), where \(n\) is a positive integer, under the empty relation?

(b) What are the maximal and minimal elements, if any, of the set, \(\mathbb{N}\), of all nonnegative integers under divisibility? Is there a minimum or maximum element?

(c) What are the minimal and maximal elements, if any, of the set of integers greater than 1 under divisibility?

(d) Describe a partially ordered set that has no minimal or maximal elements.

(e) Describe a partially ordered set that has a unique minimal element, but no minimum element. *Hint:* It will have to be infinite.

Problem 4. (a) Give an example of a digraph in which a vertex \(v\) is on a positive even-length closed walk, but no vertex is on an even-length cycle.

(b) Give an example of a digraph in which a vertex \(v\) is on an odd-length closed walk but not on an odd-length cycle.
(c) Prove that every odd-length closed walk contains a vertex that is on an odd-length cycle.

**Problem 5.**

We want to schedule \( n \) tasks with prerequisite constraints among the tasks defined by a DAG.

(a) Explain why any schedule that requires only \( p \) processors must take time at least \( \lceil n/p \rceil \).

(b) Let \( D_{n,t} \) be the DAG with \( n \) elements that consists of a chain of \( t - 1 \) elements, with the bottom element in the chain being a prerequisite of all the remaining elements as in the following figure:

![Diagram of DAG](image-url)

What is the minimum time schedule for \( D_{n,t} \)? Explain why it is unique. How many processors does it require?

(c) Write a simple formula, \( M(n, t, p) \), for the minimum time of a \( p \)-processor schedule to complete \( D_{n,t} \).

(d) Show that every partial order with \( n \) vertices and maximum chain size, \( t \), has a \( p \)-processor schedule that runs in time \( M(n, t, p) \).

*Hint: Use induction on \( t \).*

**Problem 6.**

The **weight of a walk** in a weighted graph is the sum of the weights of the successive edges in the walk. The **minimum weight matrix** for length \( k \) walks in an \( n \)-vertex graph \( G \) is the \( n \times n \) matrix \( W \) such that for \( u, v \in V(G) \),

\[
W_{uv} := \begin{cases} w & \text{if } w \text{ is the minimum weight among length } k \text{ walks from } u \text{ to } v, \\ \infty & \text{if there is no length } k \text{ walk from } u \text{ to } v. \end{cases}
\]

(1)

The \( \min^+ \) product of two \( n \times n \) matrices \( W \) and \( M \) with entries in \( \mathbb{R} \cup \{\infty\} \) is the \( n \times n \) matrix \( W \cdot M \) whose \( ij \) entry is

\[
(W \cdot M)_{ij} := \min \{W_{ik} + M_{kj} \mid 1 \leq k \leq n\}.
\]

Prove the following theorem.

**Theorem.** If \( W \) is the minimum weight matrix for length \( k \) walks in a weighted graph \( G \), and \( M \) is the minimum weight matrix for length \( m \) walks, then \( W \cdot M \) is the minimum weight matrix for length \( k + m \) walks.