Staff Solutions to Problem Set 9

Reading: Chapter 13. Asymptotics

Problem 1.
Assuming the following sum equals a polynomial in \( n \), find the polynomial. Optionally, you might want to use induction to prove that the sum equals the polynomial you find, but no such proof is required for full credit.

\[
\sum_{i=1}^{n} i^3
\]

Solution. As in Section 13.2, a sensible guess is that the sum of the first \( n \) cubes will result in a fourth-degree polynomial in \( n \):

\[
\sum_{i=1}^{n} i^3 = an^4 + bn^3 + cn^2 + dn + e.
\]

We need to determine the coefficients.

\( n = 0 \) implies \( 0 = e \).
\( n = 1 \) implies \( 1 = a + b + c + d + e \).
\( n = 2 \) implies \( 9 = 16a + 8b + 4c + 2d + e \).
\( n = 3 \) implies \( 36 = 81a + 27b + 9c + 3d + e \).
\( n = 4 \) implies \( 100 = 256a + 64b + 16c + 4d + e \).

Solving this equation gives:

\[
a = \frac{1}{4}, \quad b = \frac{1}{2}, \quad c = \frac{1}{4}, \quad d = 0, \quad e = 0,
\]

which would imply that

\[
\sum_{i=1}^{n} i^3 = \frac{n^4 + 2n^3 + n^2}{4}.
\]  \( \text{(1)} \)

We now verify (1) by induction on \( n \) with induction hypothesis \( P(n) \) given by (1).

Base case: \( n = 0 \). The left hand side of (1) is an empty sum, which equals 0 by convention. The right hand side is also 0.

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Inductive step:
\[
\sum_{i=1}^{n+1} i^3 = \left( \sum_{i=1}^{n} i^3 \right) + (n+1)^3 \\
= \frac{n^4 + 2n^3 + n^2}{4} + (n+1)^3 \\
= \frac{n^4 + 2n^3 + n^2 + 4(n^3 + 3n^2 + 3n + 1)}{4} \\
= \frac{(n^4 + 4n^3 + 6n^2 + 4n + 1) + (2n^3 + 6n^2 + 6n + 2) + (n^2 + 2n + 1)}{4} \\
= \frac{(n + 1)^4 + 2(n + 1)^3 + (n + 1)^2}{4}.
\]
This proves \( P(n+1) \), completing the induction step.

Problem 2.
Show that
\[
\ln(n^2!) = \Theta(n^2 \ln n)
\]

*Hint:* Stirling’s formula for \( (n^2)! \).

**Solution.** By Stirling’s formula:
\[
(n^2)! \sim \sqrt{2\pi n^2} \left( \frac{n^2}{e} \right)^{n^2}.
\]

We can now take logarithms (see Problem 13.34) to get:
\[
\ln(n^2!) \sim \ln \left( \sqrt{2\pi n^2} \left( \frac{n^2}{e} \right)^{n^2} \right) \\
= \ln(\sqrt{2\pi n^2}) + \ln \left( \left( \frac{n^2}{e} \right)^{n^2} \right) \\
= \frac{1}{2} \ln 2\pi + \ln n + n^2 \ln \left( \frac{n^2}{e} \right) \\
= \frac{1}{2} \ln 2\pi + \ln n + n^2(2 \ln n - 1) \\
\text{(\( \ln e = 1 \))}
\]
It is then easy to see that this expression and \( n^2 \ln n \) are big-O of each other, so we conclude that \( \ln(n^2!) = \Theta(n^2 \ln n) \).

Problem 3.
Prove that
\[
\sum_{k=1}^{n} k^6 = \Theta(n^7).
\]

*Hint:* One solution uses the Integral Method, and there are other workable approaches that avoid calculus.
Solution. Let $S_n := \sum_{k=1}^{n} k^6$. One approach is to use the Integral Method:

$$\frac{n^7}{7} = \int_{0}^{n} x^6 \, dx \leq S_n \leq \int_{0}^{n} (x + 1)^6 \, dx = \frac{(n + 1)^7}{7} - \frac{1}{7}.$$  

So we have $n^7 \leq 7S_n$, and so $n^7 = O(S_n)$. Also $(n + 1)^7 / 7 - 1 / 7 = O(n^7)$, and so $S_n = O(n^7)$. Hence, $S_n = \Theta(n^7)$.

An alternative approach not using the Integral Method goes as follows. There are $n$ terms in $S_n$ and each term is at most $n^6$, so $S_n \leq n \cdot n^6 = n^7 = O(n^7)$. So $S_n = O(n^7)$.

On the other hand, at least $(n - 1)/2$ of the terms are as large as $[(n - 1)/2]^6$, so

$$S_n \geq (\frac{n-1}{2}) \cdot [(\frac{n-1}{2})]^6$$

$$= \frac{(n-1)^7}{2^7}$$

$$\geq \frac{n^7}{3^7}$$

for $n > 3$, so $n^7 \leq 3^7 \cdot S_n$. In other words, $n^7 = O(S_n)$. 

\[\square\]