Staff Solutions to Problem Set 8

Reading: Sections 11.7–11.10, 11.6

Problem 1.
Prove Corollary 11.10.12: If all edges in a finite weighted graph have distinct weights, then the graph has a unique MST.

Hint: Suppose $M$ and $N$ were different MST’s of the same graph. Let $e$ be the smallest edge in one and not the other, say $e \in M - N$, and observe that $N + e$ must have a cycle.

Solution. Assume for the sake of contradiction that $M$ and $N$ were different MST’s of the same graph. Let $e$ be a minimum weight edge as in the hint.

Since $N$ is a spanning tree, we know that $N + e$ is connected, and it has too many edges to be a tree, so $N + e$ has a cycle. Since $M$ has no cycles, the cycle in $N + e$ cannot consist solely of edges from $M$. So there must be an edge $g \in N - M$ on the cycle, and we know that $w(g)$ must be larger than $w(e)$ by definition of $e$. Removing $g$ from $N + e$ leaves a connected graph with the same number of nodes and edges as $N$, so $N + e - g$ must be a spanning tree. But $N + e - g$ weighs $w(g) - w(e)$ less than $N$, contradicting the fact that $N$ is a minimum weight spanning tree.

Problem 2.
A basic example of a simple graph with chromatic number $n$ is the complete graph on $n$ vertices, that is $K_n$. This implies that any graph with $K_n$ as a subgraph must have chromatic number at least $n$. It’s a common misconception to think that, conversely, graphs with high chromatic number must contain a large complete subgraph. In this problem we exhibit a simple example countering this misconception, namely a graph with chromatic number four that contains no triangle—length three cycle—and hence no subgraph isomorphic to $K_n$ for $n \geq 3$. Namely, let $G$ be the 11-vertex graph of Figure 2. The reader can verify that $G$ is triangle-free.

(a) Show that $G$ is 4-colorable.

Solution. Figure 1 shows a valid coloring.

(b) Prove that $G$ can’t be colored with 3 colors.

Solution. Assume by contradiction that there is one coloring using only 3 colors: red, blue and green. The outer pentagon of the graph is an odd-length cycle, and so requires all 3 colors. So we can assume wlog that the outer pentagon is colored as shown in the left hand side of Figure 3.

This coloring of the pentagon forces the coloring of three interior points, as shown in the right hand side of Figure 3. Now the point in the center has neighbors with all three colors, so it is impossible to color it.

The graph $G$ is the known as the Groetzsch graph. It is the smallest triangle-free graph with chromatic number 4. It turns out that for any $n > 0$, there is a triangle-free graph with chromatic number $n$, see Mycielski graphs on the Wolfram Mathworld web site.
Problem 3.
The preferences among 4 boys and 4 girls are partially specified in the following table:

<table>
<thead>
<tr>
<th></th>
<th>G1</th>
<th>G2</th>
<th>–</th>
<th>–</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
<td>G1</td>
<td>G2</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>B2</td>
<td>G2</td>
<td>G1</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>B3</td>
<td>–</td>
<td>–</td>
<td>G4</td>
<td>G3</td>
</tr>
<tr>
<td>B4</td>
<td>–</td>
<td>–</td>
<td>G3</td>
<td>G4</td>
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<tr>
<td>G1</td>
<td>B2</td>
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<tr>
<td>G2</td>
<td>B1</td>
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</tr>
<tr>
<td>G3</td>
<td>–</td>
<td>–</td>
<td>B3</td>
<td>B4</td>
</tr>
<tr>
<td>G4</td>
<td>–</td>
<td>–</td>
<td>B4</td>
<td>B3</td>
</tr>
</tbody>
</table>

(a) Verify that

\[(B1, G1), (B2, G2), (B3, G3), (B4, G4)\]

will be a stable matching whatever the unspecified preferences may be.

Solution.

• B1 and B2 get their 1st choice, so won’t be in a rogue couple.
  • G1 and G2 get their 2nd choices, so won’t be in a rogue couple with the other two boys, B3 or B4. So G1 and G2 won’t be in any rogue couple, either.
  • G3 and G4 get their best remaining choices, so will never be in a rogue couple.
  • This leaves no possible rogue partners for B3 and B4.

So the marriages are sure to be stable.

Figure 2  Graph $G$ with no triangles and $\chi(G) = 4$. 
(b) Explain why the stable matching above is neither boy-optimal nor boy-pessimal and so will not be an outcome of the Mating Ritual.

Solution. Notice that giving $G_1$ and $G_2$ their first choices, that is, marrying $(B_1, G_2)$ and $(B_2, G_1)$ would also be stable for the same reason. But with this switch, $B_1$ does worse. So the stable matching above is not boy-pessimal.

Likewise, after marrying off the first two boys and girls, giving $B_3$ and $B_4$ their best remaining choices, that is, marrying $(B_3, G_4), (B_4, G_3)$, will also be stable. But with this switch, $B_3$ does better. So the stable matching above is not boy-optimal.

This implies that the stable matching above would not be produced by the Mating Ritual.

(c) Describe how to define a set of marriage preferences among $n$ boys and $n$ girls which have at least $2^{n/2}$ stable assignments.

Hint: Arrange the boys into a list of $n/2$ pairs, and likewise arrange the girls into a list of $n/2$ pairs of girls. Choose preferences so that the $k$th pair of boys ranks the $k$th pair of girls just below the previous pairs of girls, and likewise for the $k$th pair of girls. Within the $k$th pairs, make sure each boy’s first choice girl in the pair prefers the other boy in the pair.

Solution. Suppose a match has the two boys in the $k$th pair married to the two girls in the $k$th pair, for $1 \leq k \leq n/2$. A boy John, in the $k$th pair of boys will never be in a rogue couple with a girl, Jill, who is in the $j$ th pair of girls for $j \neq k$, because if $j > k$, then Jill prefers her partner in the $j$th pair to John, and if $j < k$ then John prefers his partner in the $k$th pair to Jill.

A rogue couple can only involve a boy, John, and a girl, Mary, in the same pair, but this is impossible since (exactly) one of John and Mary must be married to their preferred choice in their pair.

Since each boy can be stably married to either of the girls in the $k$th pair, and there are $n/2$ pairs, the total number of such stable matchings is $2^{n/2}$.