Staff Solutions to Problem Set 3

Reading:
- Section 4.3. Functions through 4.5. Finite Cardinality.
- Chapter 5. Induction through 5.3. Induction vs WOP.

Problem 1.
The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined as follows:

$$F_n := \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ F_{n-1} + F_{n-2} & \text{if } n > 1. \end{cases}$$

Prove, using strong induction, the following closed-form formula for $F_n$:

$$F_n = \frac{p^n - q^n}{\sqrt{5}}$$

where $p = \frac{1 + \sqrt{5}}{2}$ and $q = \frac{1 - \sqrt{5}}{2}$.

Hint: Note that $p$ and $q$ are the roots of $x^2 - x - 1 = 0$, and so $p^2 = p + 1$ and $q^2 = q + 1$.

Solution. Proof. We will proceed by strong induction on $n$. Let the induction hypothesis, $P(n)$, be that the given closed-form formula holds at $n$, that is,

$$F_n = \frac{p^n - q^n}{\sqrt{5}}.$$ 

Base case ($n = 0$): $P(0)$ is true, since

$$\frac{p^0 - q^0}{\sqrt{5}} = \frac{1 - 1}{\sqrt{5}} = 0 = F_0$$

Base Case ($n = 1$): $P(1)$ is true, since

$$\frac{p^1 - q^1}{\sqrt{5}} = \frac{p - q}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 = F_1.$$

Inductive Step ($n > 1$):
Since $0 \leq n - 1, n < n + 1$, we may assume the strong induction hypothesis that $P(n - 1)$ and $P(n)$ are both true. We will use this to prove $P(n + 1)$.

1This mind-boggling formula is known as Binet's formula. We'll explain in Chapter 15, and again in Chapter 21, how it comes about.
That is, we may assume
\[ F_{n-1} = \frac{p^{n-1} - q^{n-1}}{\sqrt{5}} \]  
(1)
\[ F_n = \frac{p^n - q^n}{\sqrt{5}} \]  
(2)
\[ . \]  
(3)

From the hint we have that \( p^2 = p + 1 \), which implies that \( p^2 p^{n-1} = (p + 1) p^{n-1} \) and so
\[ p^{n+1} = p^n + p^{n-1} \]  
(4)
Likewise \( q^2 = q + 1 \), and so
\[ q^{n+1} = q^n + q^{n-1} \]  
(5)
Subtracting (5) from (4) gives
\[ p^{n+1} - q^{n+1} = p^n - q^n + p^{n-1} - q^{n-1} \]
and dividing by \( \sqrt{5} \) yields
\[ \frac{p^{n+1} - q^{n+1}}{\sqrt{5}} = \frac{p^n - q^n}{\sqrt{5}} + \frac{p^{n-1} - q^{n-1}}{\sqrt{5}} \]
\[ = F_n + F_{n-1} \]  
(by (1) and (2))  
(6)

But \( F_{n+1} = F_n + F_{n-1} \) for \( n > 1 \) by definition, so (6) implies
\[ F_{n+1} = \frac{p^{n+1} - q^{n+1}}{\sqrt{5}}. \]

That is, \( P(n + 1) \) is true in this case as well.

We conclude by strong induction that \( P(n) \) holds for all \( n \in \mathbb{N} \).

\[ \square \]

**Problem 2.**
The Block Stacking Game\(^2\) goes as follows: You begin with a stack of \( n \) boxes. Then you make a sequence of moves. In each move, you divide one stack of boxes into two nonempty stacks. The game ends when you have \( n \) stacks, each containing a single box. You earn points for each move; in particular, if you divide one stack of height \( a + b \) into two stacks with heights \( a \) and \( b \), then you score \( ab \) points for that move. Your overall score is the sum of the points that you earn for each move. What strategy should you use to maximize your total score?

As an example, suppose that we begin with a stack of \( n = 10 \) boxes. Then the game might proceed as shown in Figure 1.

Define the potential, \( p(S) \), of a stack of blocks, \( S \), to be \( k(k - 1)/2 \) where \( k \) is the number of blocks in \( S \). Define the potential, \( p(A) \), of a set of stacks, \( A \), to be the sum of the potentials of the stacks in \( A \).

Show that for any set of stacks, \( A \), if a sequence of moves starting with \( A \) leads to another set of stacks, \( B \), then \( p(A) \geq p(B) \), and the score for this sequence of moves is \( p(A) - p(B) \).

**Hint:** Try induction on the number of moves to get from \( A \) to \( B \).

**Solution.** Proof.** The proof is by ordinary induction on the number of moves, \( n \). The induction hypothesis will be

\(^2\)Excerpted from [28] Section 5.2.4.
Stack Heights | Score
---|---
10 | 25 points
5 5 | 6
5 3 2 | 4
4 3 2 1 | 4
2 3 2 1 2 | 2
2 2 2 1 2 1 | 1
1 2 2 1 2 1 1 | 1
1 1 2 1 2 1 1 1 | 1
1 1 1 2 1 1 1 1 1 1 | 1
1 1 1 1 1 1 1 1 1 1 1 1 | 1

Total Score = 45 points

Figure 1  An example of the stacking game with \( n = 10 \) boxes. On each line, the underlined stack is divided in the next step.

\[ P(n) := \text{If } n \text{ moves from a set of stacks, } A, \text{ leads to a set } B \text{ of stacks, then } p(A) \geq p(B) \text{ and the score for these } n \text{ moves is } p(A) - p(B). \]

**Base case:** \( (n = 0) \) This means no moves have been made and \( B = A \), so it’s obvious that \( P(0) \) holds.

**Inductive step:** Assume that \( P(n) \) is true for some \( n \in \mathbb{N} \), and suppose \( A \) leads to \( B \) in \( n + 1 \) moves. This means that \( A \) leads to some set of stacks, \( A_1 \), and \( A_1 \) leads to \( B \) in \( n \) steps. So the inductive hypothesis \( P(n) \) implies that \( p(A_1) \geq p(B) \) and the score for going from \( A_1 \) to \( B \) is \( p(A_1) - p(B) \).

So all we have to do is show that the score for the single move from \( A \) to \( A_1 \) is \( p(A) - p(A_1) > 0 \). The only difference between \( A \) and \( A_1 \) is that some stack \( S \) of size \( k > 1 \) splits into two stacks of sizes \( k_1, k_2 \geq 1 \) where \( k = k_1 + k_2 \). The score for such a move is \( k_1 k_2 \). Also,

\[
p(S) = \frac{(k_1 + k_2)((k_1 + k_2) - 1)}{2} = \frac{(k_1^2 + 2k_1k_2 + k_2^2) - (k_1 + k_2)}{2},
\]

and the potential of the two stack sets is the sum of their potentials, namely,

\[
\frac{k_1(k_1 - 1) + k_2(k_2 - 1)}{2} = \frac{k_1^2 + k_2^2 - (k_1 + k_2)}{2}.
\]

So the difference between these potentials equals \( k_1 k_2 > 0 \), and this is indeed equal to the score of the move.

Problem 3.
Let \( A \), \( B \), and \( C \) be sets, and let \( f : B \to C \) and \( g : A \to B \) be functions. Let \( h : A \to C \) be the composition, \( f \circ g \), that is, \( h(x) := f(g(x)) \) for \( x \in A \). Prove or disprove the following claims:

**Hint:** Arguments based on “arrows” using Definition 4.4.2 are fine.

(a) If \( h \) is surjective, then \( f \) must be surjective.

**Solution. True.**
For all \( x \) in \( C \): Since \( h \) is surjective, there exists \( y \) in \( A \) such that \( h(y) = x \). Therefore, by definition of \( h \), \( f(g(y)) = x \), so \( x \) is in the range of \( f \).

Therefore, all of \( C \) is in the image of \( f(C) \), so \( f \) is surjective.
(b) If $h$ is surjective, then $g$ must be surjective.

**Solution.** *False.*

Suppose $A = C = \{1\}$ and $B = \{1, 2\}$. Let $f$ be such that $f(1) = f(2) = 1$, and $g$ such that $g(1) = 1$. In this case $h$ is indeed surjective, as $h(1) = 1$, but $g$ is not surjective as it doesn’t map anything to 2.

(e) If $h$ is injective, then $f$ must be injective.

**Solution.** *False.*

Taking the same example as in the previous case. $h$ is injective, because only 1 maps to 1. However, $f$ is not injective as $f(1) = f(2)$.

(d) If $h$ is injective and $f$ is total, then $g$ must be injective.

**Solution.** *True.*

For all $x$ and $y$: If $g(x) = g(y)$ then since $f$ is total, $f$ is defined on $g(x)$ and

$$h(x) = f(g(x)) = f(g(y)) = h(y),$$

so $x = y$ because $h$ is injective. This means, $g$ is injective.

Note that $g$ need not be injective when $f$ is not total (see Problem 4.25).