Staff Solutions to Problem Set 10

Reading: Chapter 14 through 14.7. Counting: Bijectons, Repetitions, and Binomial Theorem.

Problem 1.
Suppose you have seven dice—each a different color of the rainbow; otherwise the dice are standard, with faces numbered 1 to 6. A roll is a sequence specifying a value for each die in rainbow (ROYGBIV) order. For example, one roll is \(3, 1, 6, 1, 4, 5, 2\) indicating that the red die showed a 3, the orange die showed 1, the yellow 6, etc.

For the problems below, describe a bijection between the specified set of rolls and another set that is easily counted using the Product, Generalized Product, and similar rules. Then write a simple arithmetic formula, possibly involving factorials and binomial coefficients, for the size of the set of rolls. You do not need to prove that the correspondence between sets you describe is a bijection, and you do not need to simplify the expression you come up with.

For example, let \(A\) be the set of rolls where 4 dice come up showing the same number, and the other 3 dice also come up the same, but with a different number. Let \(R\) be the set of seven rainbow colors and \(S := \{1, 6\}\) be the set of dice values.

Define \(B := P_{S,2} \times R_{3}\), where \(P_{S,2}\) is the set of 2-permutations of \(S\) and \(R_{3}\) is the set of size-3 subsets of \(R\). Then define a bijection from \(A\) to \(B\) by mapping a roll in \(A\) to the sequence in \(B\) whose first element is a pair consisting of the number that came up three times followed by the number that came up four times, and whose second element is the set of colors of the three matching dice.

For example, the roll \(4, 4, 2, 4, 2, 4, 4\) maps to \((2, 4), \{\text{yellow, green, indigo}\}\) in \(B\).

Now by the Bijection rule \(|A| = |B|\), and by the Generalized Product and Subset rules,

\[
|B| = 6 \cdot 5 \cdot \binom{7}{3}.
\]

(a) For how many rolls do exactly two dice have the value 6 and the remaining five dice all have different values? Remember to describe a bijection and write a simple arithmetic formula.

Example: \((6, 2, 6, 1, 3, 4, 5)\) is a roll of this type, but \((1, 1, 2, 6, 3, 4, 5)\) and \((6, 6, 1, 2, 4, 3, 4)\) are not.

Solution. As in the example, map a roll into an element of \(B := R_{2} \times P_{5}\) where \(P_{5}\) is the set of permutations of \(\{1, \ldots, 5\}\). A roll maps to the pair whose first element is the set of colors of the two dice with value 6, and whose second element is the sequence of values of the remaining dice (in rainbow order). So \((6, 2, 6, 1, 3, 4, 5)\) above maps to \((\{\text{red, yellow}\}, (2, 1, 3, 4, 5))\). By the Product rule,

\[
|B| = \binom{7}{2} \cdot 5!.
\]

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(b) For how many rolls do two dice have the same value and the remaining five dice all have different values? Remember to describe a bijection and write a simple arithmetic formula.

Example: (4, 2, 4, 1, 3, 6, 5) is a roll of this type, but (1, 1, 2, 6, 1, 4, 5) and (6, 6, 1, 2, 4, 3, 4) are not.

**Solution.** Map a roll into a triple whose first element is in S, indicating the value of the pair of matching dice, whose second element is the set of colors of the two matching dice, and whose third element is the sequence of the remaining five dice values (in rainbow order).

So (4, 2, 4, 1, 3, 6, 5) above maps to (4, {red,yellow}, (2, 1, 3, 6, 5)). Notice that the number of choices for the third element of a triple is the number of permutations of the remaining five values, namely 5!. This mapping is a bijection, so the number of such rolls equals the number of such triples. By the Generalized Product rule, the number of such triples is

\[6 \cdot \binom{7}{2} \cdot 5!.

Alternatively, we can define a map from rolls in this part to the rolls in part (a), by replacing the value of the duplicated values with 6’s and replacing any 6 in the remaining values by the value of the duplicated pair. So the roll (4, 2, 4, 1, 3, 6, 5) would map to the roll (6, 2, 6, 1, 3, 4, 5). Now a type a roll, r, is mapped to by exactly the rolls obtainable from r by exchanging occurrences of 6’s and i’s, for i = 1, . . . , 6. So this map is 6-to-1, and by the Division rule, the number of rolls here is 6 times the number of rolls in part (a).

(c) For how many rolls do two dice have one value, two different dice have a second value, and the remaining three dice a third value? Remember to describe a bijection and write a simple arithmetic formula.

Example: (6, 1, 2, 1, 2, 6, 6) is a roll of this type, but (4, 4, 4, 4, 1, 3, 5) and (5, 5, 5, 6, 6, 1, 2) are not.

**Solution.** Map a roll of this kind into a 4-tuple whose first element is the set of two numbers of the two pairs of matching dice, whose second element is the set of two colors of the pair of matching dice with the smaller number, whose third element is the set of two colors of the larger of the matching pairs, and whose fourth element is the value of the remaining three dice. For example, the roll (6, 1, 2, 1, 2, 6, 6) maps to the triple

\[\{1, 2\}, \{orange,green\}, \{yellow,blue\}, 6\].

There are \(\binom{6}{2}\) possible first elements of a triple, \(\binom{7}{2}\) second elements, \(\binom{5}{2}\) third elements since the second set of two colors must be different from the first two, and 4 ways to choose the value of the three dice since their value must differ from the values of the two pairs. So by the Generalized Product rule, there are

\[\binom{6}{2} \cdot \binom{7}{2} \cdot \binom{5}{2} \cdot 4\]

possible rolls of this kind.

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**Problem 2.**

Answer the following questions with a number or a simple formula involving factorials and binomial coefficients. Briefly explain your answers.

(a) How many ways are there to order the 26 letters of the alphabet so that no two of the vowels a, e, i, o, u appear consecutively and the last letter in the ordering is not a vowel?

**Hint:** Every vowel appears to the left of a consonant.
Solution. The constraint on where vowels can appear is equivalent to the requirement that every vowel appears to the left of a consonant. So given a sequence of the 21 consonants, there are \( \binom{21}{5} \) positions where the 5 vowels can be placed. After determining such a placement, we can reorder the consonants and vowels in any order. Thus, the number is:

\[ \binom{21}{5} \cdot 21! \cdot 5! . \]

(b) How many ways are there to order the 26 letters of the alphabet so that there are at least two consonants immediately following each vowel?

Solution. The pattern of consonants and vowels in any permutation of the 26 letters of the alphabet can be indicated by a binary string with 5 ones indicating where the vowels occur and 21 zeros where the consonants occur. Patterns where every vowel has at least two consonants to its right can be constructed by taking a sequence of 16 zeros and inserting “10” to the left of 5 of the 16 zeros. There are \( \binom{16}{5} \) ways to do this. For any such pattern, there are 5! ways to place the vowels in the positions where ones occur and 21! ways to place the consonants where the zeroes occur. Thus, the final answer is:

\[ \binom{16}{5} \cdot 5! \cdot 21! . \]

(e) In how many different ways can \( 2n \) students be paired up?

Solution.

\[ \frac{(2n)!}{n!2^n} . \]  

There are \( (2n)! \) permutations of the \( 2n \) people. A permutation can be mapped to a pairing up of the \( 2n \) people by pairing consecutive people in the permutation. That is, one pair consists of the first and second people, another pair of the third and fourth people, through an \( n \)th pair of the \( (2n-1) \)st and \( 2n \)th people in the permutation.

Two permutations will map to the same set of pairs iff one permutation can be changed into the other permuting the order of the consecutive pairs or by switching the elements of a pair. Since there are \( n \) consecutive pairs, there are \( n! \) ways to permute the pairs and \( 2^n \) ways to switch the order within pairs. So the mapping from permutations to sets of pairs is \( n!2^n \). Now the Division Rule 14.4 implies that the number of ways to divide \( 2n \) people into \( n \) pairs is given by (1).

(d) Two \( n \)-digit sequences of digits 0,1,\ldots,9 are said to be of the same type if the digits of one are a permutation of the digits of the other. For \( n = 8 \), for example, the sequences 0308929 and 0023899 are the same type. How many types of \( n \)-digit sequences are there?

Solution. The type of a string is determined simply by the numbers of occurrences of the digits 0–9 in the string. So there is a bijection between types of strings and strings with \( n \) 0’s and nine 1’s: the length of the block of 0’s before the \( i \)th 1 (starting with \( i = 0 \)) equals the number of occurrences of the digit \( i \), and the length of the block of 0’s following the last 1 equals the number of occurrences of the digit 9. Therefore, the number of different types is

\[ \binom{n + 9}{9} . \]
Problem 3. (a) Use the Multinomial Theorem 14.6.5 to prove that
\[ (x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + x_2^p + \cdots + x_n^p \pmod{p} \] (2)
for all primes \( p \). (Do not prove it using Fermat’s “little” Theorem. The point of this problem is to offer an independent proof of Fermat’s theorem.)

Hint: Explain why \( \binom{p}{k_1, k_2, \ldots, k_n} \) is divisible by \( p \) if all the \( k_i \)'s are positive integers less than \( p \).

Solution. By the Multinomial Theorem 14.6.5, \((x_1 + x_2 + \cdots + x_n)^p\) is a sum of monomials in \( x_1, \ldots, x_n \) whose coefficients are
\[ \binom{p}{k_1, k_2, \ldots, k_n} \]
where the sum of the \( k_i \)'s is \( p \). But if all the \( k_i \)'s are less than \( p \), then none of the denominator terms divides the numerator, \( p \), and so the multinomial coefficient is divisible by \( p \). So the only coefficients not divisible by \( p \) are the coefficients of the terms \( x_i^p \), and all the other terms are \( \equiv 0 \pmod{p} \). ■

(b) Explain how (2) immediately proves Fermat’s Little Theorem 8.10.8: \( n^{p-1} \equiv 1 \pmod{p} \) when \( n \) is not a multiple of \( p \).

Solution. Let \( x_1 = x_2 = \cdots = x_n = 1 \). Then (2) implies \( n^p \equiv n \cdot 1^p = n \pmod{p} \). If \( n \) is not a multiple of \( p \), then we can then cancel \( n \) to get \( n^{p-1} \equiv 1 \pmod{p} \). ■