Staff Solutions to Midterm Exam April 15

STAFF NOTE: Topics: Digraphs, DAGs, Partial orders and equivalence, Simple Graphs, Trees, Stable matching, and Sums and Products

Problem 1 (Scheduling) (15 points).
The following DAG describes the prerequisites among tasks \{A, \ldots, H\}.

(a) What are the two maximum size antichains?

Solution.  \( A, B, C \) or \( D, B, C \)

(b) If each task takes unit time to complete, what is the minimum parallel time to complete all the tasks?

Solution.  This is the size of the maximum chain, namely, 4. We also need to check that some schedule has this length, which is easy enough to do.

(c) What is the minimum parallel time if no more than two tasks can be completed in parallel?

Solution.  Still 4: schedule \( A, B \) then \( C, D \) then \( E, F \) then \( G, H \).

Problem 2 (Partial Orders & Equivalence) (20 points).
Let \( A \) be a nonempty set.

(a) Describe a single relation on \( A \) that is both an equivalence relation and a weak partial order on \( A \).

Solution.  The equality relation, \( \text{Id}_A \), on \( A \) is obviously an equivalence. It is also vacuously antisymmetric, since there are no \( a, b \in A \) such that \( a \neq b \) AND \( a \text{ Id}_A b \).

(b) Prove that the relation of part (a) is the only relation on \( A \) with these properties.
Solution. Proof. Suppose $R$ is a relation on $A$ that is an equivalence and a wpo. Since $R$ is an equivalence, we know that $R$ is reflexive. So to show that $R = \text{Id}_A$, we need only show that

$$a \ R \ b \ \text{IMPLIES} \ a = b.$$ 

So suppose $a \ R \ b$. Since $R$ is symmetric, we can conclude that $b \ R \ a$. Now since $R$ is antisymmetric, we conclude that $a = b$. 

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Problem 3 (Simple graphs) (20 points). (a) Give an example of a simple graph that has two vertices $u \neq v$ and two distinct paths between $u$ and $v$, but no cycle including either $u$ or $v$.

*Hint:* There is an example with 5 vertices.

**Solution.** Define

$$V := \{u, v, a, b, c\},$$

$$E := \{\langle u-a \rangle, \langle a-b \rangle, \langle b-c \rangle, \langle c-a \rangle, \langle c-v \rangle\}.$$ 

Two paths from $u$ to $v$ are

$$u \langle u-a \rangle a \langle a-c \rangle c \langle c-v \rangle v$$

and

$$u \langle u-a \rangle a \langle a-b \rangle b \langle b-c \rangle c \langle c-v \rangle v.$$ 

(b) Prove that if there are different paths between two vertices in a simple graph, then the graph has a cycle.

**Solution.** Proof. Call two vertices $u \neq v$ a *different-path-pair* (dpp) if there are distinct paths between them. Suppose $u, v$ is a dpp whose distance is minimum among all dpp's (we apply the WOP implicitly!), and let $p$ be a shortest path between $u$ and $v$. By definition of dpp, there must be another path $q \neq p$ between $u$ and $v$.

We claim that, other than $u$ and $v$, there cannot be a vertex that appears in both paths $p$ and $q$. This implies that $q \ 	ext{reverse}(p)$ is a cycle.

So we just have to prove the claim: suppose to the contrary there was such a vertex, $w$, appearing in both $p$ and $q$. This means that

$$p = p_1 \hspace{0.1cm} \overleftarrow{w} \hspace{0.1cm} p_2$$

and

$$q = q_1 \hspace{0.1cm} \overleftarrow{w} \hspace{0.1cm} q_2$$

for some walks $p_1, q_1$ that start at $u$ and end at $w$, and walks $p_2, q_2$ that start at $w$ and end at $v$. But since $p \neq q$, either $p_1 \neq q_1$ or $p_2 \neq q_2$, which implies that either $u, w$ is a dpp or $w, v$ is a dpp, and this dpp will have a shorter path between them than $u, v$. This contradicts the fact that among all dpp’s, $u, v$ have a shortest length path between them. So the claim must be true.

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Note that the proof above is a rephrasing of the proof that a simple graph is a tree iff there is a unique path between any two vertices, Theorem 11.10.3.2. In fact, appeal to this Theorem yields an immediate proof of this part: if $u, v$ are a dpp, then by the Theorem, the connected component containing $u, v$ is not a tree, which means it contains a cycle by definition.
Problem 4 (Trees & Coloring) (20 points).
Prove by induction that, using a fixed set of \( n > 1 \) colors, there are exactly \( n \cdot (n-1)^{m-1} \) different colorings\(^1\) of any tree with \( m \) vertices.

Solution. Proof. By induction on the number of vertices, \( m \).

Induction hypothesis: \( P(m) \): For all \( m \)-vertex trees, \( T \), there are \( n \cdot (n-1)^{m-1} \) different colorings of \( T \).

Base case: \( (m = 1) \). There are \( n = n(n-1)^{1-1} \) ways to color one vertex.

Induction step: Let \( T \) be a tree with \( m + 1 \) vertices for some \( m \geq 1 \). Let \( v \) be a leaf of \( T \). Then removing \( v \) and its incident edge, we obtain a tree \( T - v \) with \( m \) vertices. We may assume by induction that there are \( n \cdot (n-1)^{m-1} \) colorings of \( T - v \). For each such coloring of \( T - v \), there are \( n - 1 \) ways to assign a color to \( v \) to obtain an coloring of \( T \), so there are \( n \cdot (n-1)^{m-1} \cdot (n - 1) = n \cdot (n-1)^{m} \) colorings of \( T \), which proves \( P(m + 1) \).

Conclusion: Thus, there are exactly \( n \cdot (n-1)^{m-1} \) different colorings of any tree with \( m \) vertices. \( \blacksquare \)

Problem 5 (Stable Marriage) (15 points).
The Mating Ritual for finding stable marriages works without change when there are at least as many, and possibly more, men than women. You may assume this. So the Ritual ends with all the women married and no rogue couples for these marriages, where an unmarried man and a married woman who prefers him to her spouse is also considered to be a “rogue couple.”

Let Alice be one of the women, and Bob be one of the men. Circle the properties below that are preserved invariants of the Mating Ritual when there are at least as many men as women.

(a) Alice has a suitor (man who is serenading her) whom she prefers to Bob.

Solution. Invariant: Alice’s suitors can only improve during the Ritual. \( \blacksquare \)

(b) Alice is the only woman on Bob’s list.

Solution. Not preserved. It would be invariant if there were at least as many women as men, since Bob must marry Alice in that case. But if there are more men, Bob may wind up unmarried. \( \blacksquare \)

(c) Alice has no suitor.

Solution. Not preserved; Alice may not have a suitor on the first day—if, for example, she’s not at the top of any man’s list—but may get a suitor after the first round of rejections. \( \blacksquare \)

(d) Bob prefers Alice to the women he is serenading.

Solution. Invariant. Bob works down his list, so if Alice is crossed off, Bob preferred her to anybody left on his list. \( \blacksquare \)

(e) Bob is serenading Alice.

Solution. Not invariant: If Bob serenades Alice and gets rejected by her, he will stop serenading her. \( \blacksquare \)

(f) Bob is not serenading Alice.

\(^1\)That is, an assignment of colors to vertices so that no two adjacent vertices are assigned the same color.
Solution. Not invariant: Bob might serenade someone else, get rejected by her, and then serenade Alice next. ■

(g) Bob’s list of women to serenade is empty.

Solution. Invariant: No woman will ever get added to Bob’s list, so once his list is empty, it stays empty. (If we have at least as many women as men, Bob’s list will never be empty, since the Ritual guarantees he will be married in the end. But this predicate is still an invariant because it is always false.) ■

Problem 6 (Sums & Integrals) (10 points).
There is a number $a$ such that

$$\sum_{i=1}^{\infty} i^p$$

converges to a finite value iff $p < a$.

(a) What is the value of $a$?

Solution. $-1$. ■

(b) Circle all of the following that would be good approaches as part of a proof that this value of $a$ is correct.

i. Find a closed form for $\int_1^{\infty} x^p \, dx$.

ii. Find a closed form for $\int_1^{\infty} i^p \, dp$.

iii. Induction on $p$.

iv. Induction on $n$ using the following sum

$$\sum_{i=1}^{n} i^p.$$  

v. Compare the series term-by-term with the Harmonic series.

Solution. (1) and (5).

(1) is correct because $x^p$ is decreasing in $x$ if $p < 0$ and increasing if $p \geq 0$. So by Theorem 13.3.2, the sum can be approximated by the integral

$$\int_1^{\infty} x^p \, dx = \begin{cases} \frac{-1}{p+1} & \text{if } p < -1, \\ \infty & \text{if } p \geq -1. \end{cases}$$

So if $p < -1$, Theorem 13.3.2 implies that the sum is bounded above by one plus the integral. Since the sum is increasing, this implies it has a finite limit, that is, it converges.

Likewise, the sum is bounded below by the integral, and so diverges if $p \geq -1$.

(5) is correct because for $p = -1$, the sum is the harmonic series which we know diverges. But for $p \geq -1$, the value of $i^p$ increases with $p$, the sum will be larger, and hence also diverge for $p > -1$.

Induction on $n$ is not a plausible approach because ideas from the other approaches would be needed to handle the induction step anyway, at which point the induction would be moot. ■