Staff Solutions to Final Exam

Problem 1 (Probable Satisfiability) (6 points).
Truth values for propositional variables \(P, Q, R\) are chosen independently, with

\[
\Pr [P = T] = 1/2, \ \Pr [Q = T] = 1/3, \ \Pr [R = T] = 1/5.
\]

What is the probability that the formula

\[(P \text{ implies } Q) \text{ implies } R\]  

is true?

Solution. Consider the separate cases in which (1) is true:

- \(R = T\): This case happens with probability \(1/5\).
- \(R = F \text{ AND } Q = F \text{ AND } P = T\): This case happens with probability \((4/5)(2/3)(1/2)\).
  
  So the probability (1) is true is
  
  \[
  \frac{1}{5} + \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{7}{15}.
  \]

  Another breakdown into cases is \(P = T \text{ AND } Q = F\), and \(\text{NOT}(P = T \text{ AND } Q = F) \text{ AND } R = T\) with probability
  
  \[
  \frac{1}{2} \cdot \frac{2}{3} + (1 - \frac{2}{3}) \cdot \frac{1}{5} = \frac{7}{15}
  \]

Problem 2 (Induction, Trees) (10 points).
A simple graph, \(G\), is said to have width 1 iff there is a way to list all its vertices so that each vertex is adjacent to at most one vertex that appears earlier in the list.

Prove that every finite tree has width one.

Solution. By induction on the number of vertices, \(n\). The induction hypothesis is

\(Q(n) := \) all \(n\)-vertex trees \(T\) have width one.

Base case: \((n = 1)\). Trivial.

Induction step. Assume that that \(Q(n)\) is true for some \(n \geq 1\) and let \(T\) be an \((n + 1)\)-vertex tree. We need only show that \(T\) has width one.

By Theorem 12.9.3, every tree with at least two vertices has a leaf. Let \(v\) be a leaf of \(T\). Then \(T - v\) has width one by Induction Hypothesis, so its vertices can be listed with each vertex adjacent to at most one vertex earlier in the list. Since \(v\) has degree one, we can add it to the end of the list of vertices for \(T - v\) to obtain the required list for \(T\). Hence \(T\) has width one, as claimed.

This proves \(Q(n + 1)\) and completes the induction step.■
Problem 3 (Number Theory) (8 points).
Indicate whether the following statements are true or false. For each of the false statements, give counterexamples. All variables range over the integers, \( \mathbb{Z} \).

(a) For all \( a \) and \( b \), there are \( x \) and \( y \) such that: \( ax + by = 1 \).

Solution. FALSE. \( a \) and \( b \) must be relatively prime. \( a = b = 2 \) is a counterexample.

(b) \( \gcd(mb + r, b) = \gcd(r, b) \) for all \( m, r \) and \( b \).

Solution. TRUE.

(c) \( k^{p−1} \equiv 1 \pmod{p} \) for every prime \( p \) and every \( k \).

Solution. FALSE. \( k \) must be relatively prime to \( p \). \( k = p = 2 \) is a counterexample.

(d) For primes \( p \neq q \), \( \phi(pq) = (p−1)(q−1) \), where \( \phi \) is Euler's totient function.

Solution. TRUE.

(e) If \( a \) and \( b \) are relatively prime to \( d \), then

\[ [ac \equiv bc \pmod{d}] \quad \text{IMPLIES} \quad [a \equiv b \pmod{d}]. \]

Solution. FALSE. To cancel \( c \), we need that \( c \) is relatively prime to \( d \).

A counterexample is \( a = 1, b = 2 \) and \( d = 3 \), and \( c = 0 \).

Problem 4 (Scheduling & DAGs) (8 points).
Sauron finds that conquering Middle Earth breaks down into a bunch of tasks. Each task can be completed by a horrible creature called a Ringwraith in exactly one week. Sauron realizes the prerequisite structure among the tasks defines a partial order. He has \( n \) tasks in his partial order, with a maximum length chain of \( t \) tasks.

In order to complete all \( n \) tasks in \( t \) weeks, Sauron will need to have crew of Ringwraiths working in parallel. In answering the following questions, do not make any assumptions about the values of \( n \) and \( t \) besides \( 1 \leq t \leq n \).

(a) Write a simple formula involving \( n \) and \( t \) for the smallest number of Ringwraiths that Sauron could possibly need.

Solution.

\[ \lceil \frac{n}{t} \rceil \]

Each of the \( n \) tasks must be completed in one of the \( t \) weeks. Thus, by the pigeonhole principle, at least \( \lceil n/t \rceil \) must be completed in some week, thus requiring at least this many Ringwraiths.

This bound can’t be made larger, since a partial order with a single length \( t \) chain and the other \( n−t \) tasks with no prerequisites can be completed in time \( t \) by doing \( \lceil n/t \rceil \) each week.

(b) On the other hand, if Sauron is unlucky, he may need a crew of Ringwraiths as large as \( n−t+1 \) in order to conquer Middle Earth in \( t \) weeks. Describe a partial order with \( n \) tasks and maximum length chain of \( t \) that would require this many Ringwraiths.
Solution. There will be \( n - t + 1 \) Ringwraiths needed if there is an antichain of \( n - t + 1 \) tasks that cannot be started until a chain of \( t - 1 \) tasks is completed, as in the following figure:

![Diagram of a tree structure with \( t - 1 \) as the root and \( n - (t - 1) \) as the leaf nodes, illustrating the task dependency graph.]

Problem 5 (Simple Graphs) (8 points).

The degree sequence of a simple graph \( G \) with \( n \) vertices is the length-\( n \) sequence of the degrees of the vertices listed in weakly increasing order. For example, if \( G \) is a 3-vertex line graph, then its degree sequence is \( (1,1,2) \). On the other hand, \( (0,0,2) \) is not a degree sequence, since in any graph with an edge, there are at least two vertices of positive degree, namely, the endpoints of the edge.

Briefly explain why each of the following sequences is not a degree sequence of any connected simple graph.

(a) \( (1,2,3,4,5,6,7) \)

Solution. There are only 7 vertices, so the degree of any vertex is at most 6.

(b) \( (1,3,3,4,4,4) \)

Solution. By the Handshaking Lemma, the sum of degrees in any simple graph must be even, which is not true in this case since \( 1 + 3 + 3 + 4 + 4 + 4 = 19 \).

(c) \( (1,1,1,1) \)

Solution. This can only be two disjoint edges, so the graph is not connected.

(d) \( (1,2,3,4,4) \)

Solution. There are five vertices, two of which have degree 4. So both degree-4 vertices have to be connected to all the other vertices. That means that the degree of every vertex is greater than one, which is violated in this case.
Problem 6 (Big Oh) (6 points).
Define two functions $f, g$ that are incomparable under big Oh:

$$f \notin O(g) \text{ AND } g \notin O(f).$$

Solution. One example is,

$$f(n) := \begin{cases} n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases} \quad g(n) := \begin{cases} 0 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even,} \end{cases}$$

which can also be described by the formulas

$$f(n) := n \sin^2 \left( \frac{n\pi}{2} \right), \quad g(n) := n \cos^2 \left( \frac{n\pi}{2} \right).$$

Some students also noticed that a corner case in the book’s limit-based definition of big Oh allows $f(x) = g(x) = 0$. We’re going to try to revise the book to avoid that pathology in the future!  

Problem 7 (Counting) (10 points).
In a standard 52-card deck (13 ranks and 4 suits), a hand is a 5-card subset of the set of 52 cards. Express the answer to each part as a formula using factorial, binomial, or multinomial notation.

(a) Let $H_{NP}$ be the set of all hands that include no pairs; that is, no two cards in the hand have the same rank.

What is $|H_{NP}|$?

Solution. $|H_{NP}| = \binom{13}{5} \binom{4}{1}^5$

(b) Let $H_S$ be the set of all hands that are straights, i.e. the ranks of the five cards are consecutive. The order of the ranks is $(A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, k, A)$; note that $A$ appears twice.

What is $|H_S|$?

Solution. $|H_S| = \binom{10}{4} \binom{4}{1}^5$

(c) Let $H_F$ be the set of all hands that are flushes; that is, the suits of the five cards are identical.

What is $|H_F|$?

Solution. $|H_F| = \binom{13}{5} \binom{4}{1}$

(d) Let $H_{SF}$ be the set of all straight flush hands; that is, the hand is both a straight and a flush.

What is $|H_{SF}|$?

Solution. $|H_{SF}| = \binom{10}{4} \binom{4}{1}$

(e) Let $H_{HC}$ be the set of all high-card hands; that is, hands that do not include pairs, are not straights, and are not flushes.

What is $|H_{HC}|$?
Problem 8 (Conditional Probability) (10 points).

Here’s a variation of Monty Hall’s game: the contestant still picks one of three doors, with a prize randomly placed behind one door and goats behind the other two. But now, instead of always opening a door to reveal a goat, Monty instructs Carol to *randomly* open one of the two doors that the contestant hasn’t picked. This means she may reveal a goat, or she may reveal the prize. If she reveals the prize, then the entire game is *restarted*, that is, the prize is again randomly placed behind some door, the contestant again picks a door, and so on until Carol finally picks a door with a goat behind it. Then the contestant can choose to *stick* with his original choice of door or *switch* to the other unopened door. He wins if the prize is behind the door he finally chooses.

To analyze this setup, we define two events:

- **GP**: The event that the contestant guesses the door with the prize behind it on his first guess.
- **OP**: The event that the game is restarted at least once. Another way to describe this is as the event that the door Carol first opens has a prize behind it.

Give the values of the following probabilities:

(a) \( \Pr[OP \mid \overline{GP}] \)

**Solution.** 1/2.

(b) \( \Pr[OP] \)

**Solution.** 1/3.

\[
\Pr[OP] = \Pr[OP \mid GP] \Pr[GP] + \Pr[OP \mid \overline{GP}] \Pr[\overline{GP}] = 0 \cdot 1/3 + 1/2 \cdot 2/3 = 1/3.
\]

(c) the probability that the game will continue forever

**Solution.** 0.

There is a \((\Pr[OP])^n\) probability that the game will restart at least \(n\) times. This probability goes to zero as \(n\) goes to infinity.
(d) When Carol finally picks the goat, the contestant has the choice of sticking or switching. Let’s say that the contestant adopts the strategy of sticking. Let \( W \) be the event that the contestant wins with this strategy, and let \( w := \Pr[W] \). Express the following conditional probabilities as simple closed forms in terms of \( w \).

i) \( \Pr\left[W \mid GP\right] \)

Solution. 1.

ii) \( \Pr\left[W \mid \overline{GP} \cap OP\right] \)

Solution. \( w \)

iii) \( \Pr\left[W \mid \overline{GP} \cap \overline{OP}\right] \)

Solution. 0

(e) What is the value of \( \Pr[\overline{W}] \)?

Solution. 1/2, because

\[
\Pr[\overline{W}] = \Pr[W \mid GP] \Pr[GP] + \Pr[W \mid \overline{GP} \cap OP] \Pr[\overline{GP} \cap OP]
\]

\[
= 1 \cdot 1/3 + w \cdot 2/3 \cdot 1/2 + 0 \cdot 2/3 \cdot 1/2
\]

\[
= 1/3 + w/3.
\]

So \( w(1 - 1/3) = 1/3 \).

(f) Let \( R \) be the number of times the game is restarted before Carol picks a goat.

What is \( \text{Ex}[R] \)?

(You may express the answer as a simple closed form in terms of \( p := \Pr[OP] \).)

Solution. \( 1/(1 - p) - 1 = 3/2 - 1 = 1/2 \).

Think of not having to restart as a failure. So \( \Pr[\text{failure}] = 1 - p \), and \( \text{Ex}[R] \) is mean time to failure minus one—because we are only counting the number of “successes”—namely, \( 1/\Pr[\text{failure}] - 1 \).

Problem 9 (Expectation) (6 points).

A simple graph with \( n \) vertices is constructed by randomly placing an edge between every two vertices with probability \( p \). These random edge placements are performed independently.

(a) What is the probability that a given vertex of the graph has degree two?

Solution.

\[
\binom{n - 1}{2} \cdot p^2 \cdot (1 - p)^{n-3}
\]

The probability that the given vertex is adjacent to exactly two other given vertices and no other vertices, is \( p^2 \cdot (1 - p)^{n-3} \). There are \( \binom{n-1}{2} \) possible such sets of two other vertices.

(b) What is the expected number of nodes with degree two? (You may express your answer in terms of \( t \), where \( t \) is the answer to part (a).)
Solution. \(nt\), by linearity of expectations, because the expectation of the indicator variable for a given vertex having degree two is \(t\).

Problem 10 (Variance, Sums) (10 points).
You have a coin with probability \(p\) of flipping heads. For your first try, you flip it once. For your second try, you independently flip it twice. You continue until the \(n\)th try, where you independently flip it \(n\) times. You win a try if you flip all heads. Let \(W\) be the number of winning tries. Write a closed-form expression for \(\text{Var}[W]\).

Solution. \(W = \sum_{k=1}^{n} H_k\) where \(H_k\) is indicator winning the \(k\)th try.
Since \(\Pr[H_k = 1] = p^k\), we have \(\text{Var}[H_k] = p^k(1 - p^k)\).
Variances add because of mutual independence, so
\[
\text{Var}[W] = \sum_{k=1}^{n} p^k(1 - p^k)
= \sum_{k=1}^{n} p^k - p^{2k}
= p \sum_{k=0}^{n-1} p^k - p^2 \left( \sum_{k=0}^{n-1} (p^2)^k \right)
= p \frac{1 - p^n}{1 - p} - p^2 \frac{1 - p^{2n}}{1 - p^2}.
\]

Problem 11 (Markov & Chebyshev) (12 points).
Albert has a gambling problem. He plays 35 hands of draw poker, 30 hands of black jack, and 20 hands of stud poker per day. He wins a hand of draw poker with probability 1/7, a hand of black jack with probability 1/6, and a hand of stud poker with probability 1/5. Let \(W\) be the expected number of hands that Albert wins in a day.

(a) What is \(\text{Ex}[W]\)?

Solution. The expectation is the sum of the expectations of all the individual hands, namely
\[
35(1/7) + 30(1/6) + 20(1/5) = 14.
\]

(b) What would the Markov bound be on the probability that Albert will win at least 45 hands on a given day?

Solution. The expected number of games won is 14, so by Markov, \(\Pr[W \geq 45] \leq 14/45\).
(e) Assume the outcomes of the card games are pairwise independent. What is \( \text{Var}[W] \)? You may answer with a numerical expression that is not completely evaluated.

**Solution.** With pairwise independence, the variance can also be calculated using linearity. For an individual hand the variance is \( p(1-p) \) where \( p \) is the probability of winning. Therefore the variance is

\[
35(1/7)(6/7) + 30(1/6)(5/6) + 20(1/5)(4/5) = 2447/210 = 11 \frac{137}{210} \approx 11.7
\]

(d) What would the Chebyshev bound be on the probability that Albert will win at least 45 hands on a given day? You may answer with a numerical expression that includes the constant \( v = \text{Var}[W] \).

**Solution.**

\[
\Pr[W \geq 45] = \Pr[W - 15 \geq 30] \leq \Pr[|W - 15| \geq 30] \leq \frac{v}{31^2} = \frac{2447}{210(31)^2} \approx 0.012.
\]

(A very slightly better bound comes from using the one-sided Chebyshev bound from Problem 20.16.)

**Problem 12 (Random Walks) (6 points).**

Give simple examples of random walk graphs with the following properties.

(a) A graph with an uncountable number of stationary distributions.

**Solution.** A graph with only two vertices, 1 and 2, and only two edges \((1 \to 1)\) and \((2 \to 2)\).

(b) A graph with unique stationary distribution that is not strongly connected.

**Solution.** A graph with only two vertices, 1 and 2, and only two edges \((1 \to 2)\) and \((2 \to 2)\).

(c) A strongly connected graph with an initial distribution that does not converge to the stationary distribution.

![Figure 1](image)

**Solution.**