Staff Solutions to In-Class Problems Week 8, Wed.

STAFF NOTE: Simple Graphs: Degrees, Isomorphism, Ch.11.1–11.4

Problem 1.
A researcher analyzing data on heterosexual sexual behavior in a group of \( m \) males and \( f \) females found that within the group, the male average number of female partners was 10% larger than the female average number of male partners.

(a) Comment on the following claim. “Since we’re assuming that each encounter involves one man and one woman, the average numbers should be the same, so the males must be exaggerating.”

Solution. The averages won’t be the same. According to equation (11.1),

\[
\text{Avg. # male partners} = \frac{|F|}{|M|} \cdot \text{Avg. # female partners}
\]  

So the averages simply reflect the relative sizes of the male and female populations. This means that the males could truthfully report a higher average if there were more females.

Of course if the males exaggerate, then their reported average could be as large as they choose to fantasize, whatever the size of the female population.

(b) For what constant \( c \) is \( m = c \cdot f \)?

Solution. By equation (1), the men’s average number of partners is \( f/m \) times the females’ average, so \( f/m = 1.1 \) which implies \( m = (1/1.1) f \) and \( c = 10/11 \).

(c) The data shows that approximately 20% of the females were virgins, while only 5% of the males were. The researcher wonders how excluding virgins from the population would change the averages. If he knew graph theory, the researcher would realize that the nonvirgin male average number of partners will be \( x (f/m) \) times the nonvirgin female average number of partners. What is \( x \)?

Solution. The male average number of partners is \( f/m \) times the female average number of partners. (According to part (b), \( f/m = 1.1 \), but this number isn’t needed here.) When virgins are excluded, the ratio of the males’ average to the females’ average will be

\[
\frac{f - .2f}{m - .05m} = \frac{.8f}{.95m} = \frac{4}{19/20} \cdot \frac{f}{m},
\]

so \( x = 80/95 = 16/19 \).

(d) For purposes of further research, it would be helpful to pair each female in the group with a unique male in the group. Explain why this is not possible.
Solution. There are more females than males, so there cannot be an injective total function from the females to the males.

Problem 2. (a) Prove that in every simple graph, there are an even number of vertices of odd degree.

STAFF NOTE: Hint: The Handshaking Lemma 11.2.1.

Solution. Proof. Partitioning the vertices into those of even degree and those of odd degree, we know

\[ \sum_{v \in V} d(v) = \sum_{d(v) \text{ is even}} d(v) + \sum_{d(v) \text{ is odd}} d(v) \]

By the Handshaking Lemma, the value of the lefthand side of this equation equals twice the number of edges, and so is even. The first summand on the righthand side is even since it is a sum of even values. So the second summand on the righthand side must also be even. But since it is entirely a sum of odd values, it must must contain an even number of terms. That is, there must be an even number of vertices with odd degree.

(b) Conclude that at a party where some people shake hands, the number of people who shake hands an odd number of times is an even number.

Solution. We can represent the people at the party by the vertices of a graph. If two people shake hands, then there is an edge between the corresponding vertices. So the degree of a vertex is the number of handshakes the corresponding person performed. The result in the first part of this problem now implies that there are an even number of odd-degree vertices, which translates into an even number of people who shook an odd number of hands.

(c) Call a sequence of people at the party a handshake sequence if each person in the sequence has shaken hands with the next person, if any, in the sequence.

Suppose George was at the party and has shaken hands with an odd number of people. Explain why, starting with George, there must be a handshake sequence ending with a different person who has shaken an odd number of hands.

STAFF NOTE: Hint: Just look at all the people who appear in handshake sequences that start with George.

Solution. The handshake graph restricted to just the people who appear in handshake sequences that start with George is a subgraph of the graph of everyone at the party, and the degree of a person in the subgraph is the same in the subgraph as in the graph with everyone.

So by part (b), the subgraph must have an even number of people who shake an odd number of hands. In particular, there must be at least one other person besides George, call him Harry, who has also shaken an odd number of hands. So the handshake sequence from George that ends with Harry is what we were looking for.

Problem 3.
List all the isomorphisms between the two graphs given in Figure 1. Explain why there are no others.
Solution. These are the vertex correspondences for the four isomorphisms:

1. $A$, $B$, $C$, $D$, $E$, $F$
2. $A$, $B$, $D$, $C$, $F$, $E$
3. $B$, $A$, $C$, $D$, $E$, $F$

Some simple reasoning leads us to this answer. The first graph in this problem has exactly two nodes with degree 3 (nodes 3 and 4), as does the second graph ($c$ and $d$). Recall that nodes related by an isomorphism must have equal degrees. Thus, the isomorphism must map 3 to either $c$ or $d$, and it must map 4 to the other. Independently of this choice, nodes 1, 2, $a$, and $b$ are the only nodes connected to exactly the degree-3 nodes, so the isomorphism must divide 1 and 2 between $a$ and $b$. Node 5 is the only unassigned node connected to node 3, so 5 must be mapped to either $e$ or $f$, depending on which is adjacent to the counterpart of node 3; and now node 6 maps to whichever node is left.

Problem 4.
Which of the items below are simple-graph properties preserved under isomorphism?
(a) The vertices can be numbered 1 through 7.
(b) There is a cycle that includes all the vertices.
STAFF NOTE: If asked, explain that simple graph cycles can be defined in the essentially same way as for digraphs. The only difference is that going back and forth on the same edge—a length 2 “cycle”—is not considered to be a cycle.
(c) There are two degree 8 vertices.
(d) Two edges are of equal length.
STAFF NOTE: Not a property of simple graphs since edges don’t have length.
(e) No matter which edge is removed, there is a path between any two vertices.
(f) There are two cycles that do not share any vertices.
(g) One vertex is a subset of another one.
STAFF NOTE: If need be, remind students that we used sets as vertices when we proved that every poset can be represented sets under containment.
(h) The graph can be pictured in a way that all the edges have the same length.
(i) The OR of two properties that are preserved under isomorphism.

**STAFF NOTE:** When students have figured out that item (i) is in the positive category, ask them to do a careful proof of that fact. See solution for an example.

(j) The negation of a property that is preserved under isomorphism.

**Solution.** Item (d) is not a property of simple graphs, since edges don’t have length.

Item (g) is not preserved under isomorphism. Although it can be useful to use sets as vertices—as was done for representing DAGs in Theorem 9.7.3—vertices need not be represented by sets, and since isomorphism does not depend on what vertices are made of, vertices being sets and any set-theoretic properties they may have are not going to be preserved.

All the others are preserved. We’ll prove this just for item (i):

**Proof.** Suppose $P$ and $Q$ are graph properties preserved under isomorphism, and $G$ and $H$ are isomorphic simple graphs. Let $R := P \text{ OR } Q$. Then

$$
R(G) \text{ IFF } P(G) \text{ OR } Q(G) \text{ (def of } R) \\
\text{IMPLIES } P(H) \text{ OR } Q(H) \text{ (since } P, Q \text{ are preserved)} \\
\text{IFF } R(H) \text{ (def of } R).
$$

so $R$ is preserved, as claimed.

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**Supplemental problem**

**Problem 5.**
Let $G$ be a digraph. The neighbors of a vertex $v$ are the endpoints of the edges out of $v$. Since a digraph is formally the same as a binary relation on $V(G)$, the set of neighbors of $v$ is simply the image, $G(v)$, of $v$ under the relation $G$.

(a) Suppose $f$ is an isomorphism from $G$ to another digraph $H$. Prove that

$$
f(G(v)) = H(f(v)).$$

Your proof should follow by simple reasoning using the definitions of isomorphism and image of a vertex under the edge relation—no pictures or handwaving.

**Hint:** Prove by a chain of iff’s that

$$
h \in H(f(v)) \text{ IFF } h \in f(G(v))
$$

for every $h \in V(H)$.

**STAFF NOTE:** **Hint:** Use the fact that $h = f(u)$ for some $u \in V(G)$.

**Solution.** **Proof.** Suppose $h \in V(H)$. By definition of isomorphism, there is a unique $u \in V(G)$ such that $f(u) = h$. Then

$$
h \in H(f(v)) \text{ IFF } (f(v) \rightarrow h) \in E(H) \text{ (def of } H(f(v))) \\
\text{IFF } (f(v) \rightarrow f(u)) \in E(H) \text{ (def of } u) \\
\text{IFF } (v \rightarrow u) \in E(G) \text{ (since } f \text{ is an isomorphism)} \\
\text{IFF } u \in G(v) \text{ (def of } G(v)) \\
\text{IFF } f(u) \in f(G(v)) \text{ (def of } f\text{-image)} \\
\text{IFF } h \in f(G(v)) \text{ (def of } u)$$


So $H(f(v))$ and $f(G(v))$ have the same members and therefore are equal.

(b) Conclude that if $G$ and $H$ are isomorphic graphs, then they have the same number of vertices of out-degree $k$, for all $k \in \mathbb{N}$.

**STAFF NOTE:** Hint: $\text{outdeg}(v) := |G(v)|$.

**Solution.** Since an isomorphism is a bijection, any set of vertices and its image under an isomorphism will be the same size (Bijection Mapping Rule 4.7). So part (a) implies that an isomorphism, $f$, maps out-degree $k$ vertices to out-degree $k$ vertices. This means that the image under $f$ of the set of out-degree $k$ vertices of $G$ is precisely the set of out-degree $k$ vertices of $H$. So by the Mapping Rule again, there are the same number of out-degree $k$ vertices in $G$ and $H$. 