Staff Solutions to In-Class Problems Week 3, Fri.

Problem 1.
The inverse, $R^{-1}$, of a binary relation, $R$, from $A$ to $B$, is the relation from $B$ to $A$ defined by:

$$b R^{-1} a \iff a R b.$$  

In other words, you get the diagram for $R^{-1}$ from $R$ by “reversing the arrows” in the diagram describing $R$. Now many of the relational properties of $R$ correspond to different properties of $R^{-1}$. For example, $R$ is total iff $R^{-1}$ is a surjection.

Fill in the remaining entries is this table:

<table>
<thead>
<tr>
<th>$R$ is</th>
<th>iff $R^{-1}$ is</th>
</tr>
</thead>
<tbody>
<tr>
<td>total</td>
<td>a surjection</td>
</tr>
<tr>
<td>a function</td>
<td></td>
</tr>
<tr>
<td>a surjection</td>
<td></td>
</tr>
<tr>
<td>an injection</td>
<td></td>
</tr>
<tr>
<td>a bijection</td>
<td></td>
</tr>
</tbody>
</table>

*Hint:* Explain what’s going on in terms of “arrows” from $A$ to $B$ in the diagram for $R$.

Solution.

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The first line of the table follows from the fact that total means $[\geq 1 \text{ out}]$, so reversing the arrows gives $[\geq 1 \text{ in}]$ which is the definition of surjection.

The second line follows from the fact that function means $[\leq 1 \text{ out}]$, so reversing the arrows gives $[\leq 1 \text{ in}]$ which is the definition of injection.

The third and fourth lines follow respectively from the first and second lines.

The fifth line follows from the fact that bijection means $[= 1 \text{ out}, = 1 \text{ in}]$, so reversing the arrows gives $[= 1 \text{ in}, = 1 \text{ out}]$ which is the same.
Arrow Properties

Definition. A binary relation, $R$ is
- is a function when it has the $[\leq 1 \text{ arrow out}]$ property.
- is surjective when it has the $[\geq 1 \text{ arrows in}]$ property. That is, every point in the righthand, codomain column has at least one arrow pointing to it.
- is total when it has the $[\geq 1 \text{ arrows out}]$ property.
- is injective when it has the $[\leq 1 \text{ arrow in}]$ property.
- is bijective when it has both the $[1 \text{ arrow out}]$ and the $[1 \text{ arrow in}]$ property.

Problem 2.
Let $A = \{a_0, a_1, \ldots, a_{n-1}\}$ be a set of size $n$, and $B = \{b_0, b_1, \ldots, b_{m-1}\}$ a set of size $m$. Prove that $|A \times B| = mn$ by defining a simple bijection from $A \times B$ to the nonnegative integers from $0$ to $mn - 1$.

STAFF NOTE: Point out the Computer Science connection: this is how a compiler computes locations in memory of a 2D-array. It also indicates why most programming languages require the programmer to specify the array dimensions in advance.

Solution. Let $\{0..mn\} := \{0, 1, \ldots, mn - 1\}$. Define a mapping $f : A \times B \rightarrow [0..mn]$ by the rule
$$f(a_j, b_k) := jm + k.$$  

We will prove that $f$ is a bijection. First, observe that $f$ is a total function with domain $A \times B$. This is true since $f$ maps every pair $(a_j, b_k)$ to a nonnegative integer $jm + k$. Therefore, $f$ has the $[= 1 \text{ out}]$ property. The range of $f$ is a subset of $[0..mn]$ because for $0 \leq j \leq n - 1$ and $0 \leq k \leq m - 1$,
$$0 \leq jm + k \leq (n - 1)m + m - 1 = mn - 1.$$  

Further, observe that $f$ is a surjection (i.e., it has the $[\geq 1 \text{ in}]$ property.) This follows due to the fact that every element of $i \in [0..mn]$ is $f(a_j, b_k)$ for at least one pair $(j, k)$, namely, the $j$ is the quotient and $k$ is the remainder of $i$ divided by $m$.

In fact, the quotient-remainder pair $(j, k)$ is unique for every $i$. The proof is by contradiction. Assume the contrary, i.e., suppose that $i = jm + k = j'm + k'$, where $k \neq k'$. Without loss of generality, suppose that $k > k'$. Then, $j < j'$ (why?) and hence, $j' - j \geq 1$. Therefore, we get $k - k' = m(j' - j)$. The left hand side is positive and strictly smaller than $m$, while the right hand side is at least $m$.

From this uniqueness, it follows $f$ has the $[= 1 \text{ in}]$ property as well. Therefore, $f$ is a bijection.

STAFF NOTE: From Spring14: The next problem is the same as TP.3.6. Offer it for review, but skip if students report knowing it.

Problem 3.
Assume $f : A \rightarrow B$ is total function, and $A$ is finite. Replace the $\star$ with one of $\leq, =, \geq$ to produce the strongest correct version of the following statements:

(a) $|f(A)| \star |B|$. 

Solution. ≤, since $f(A) \subset B$.

(b) If $f$ is a surjection, then $|A| \ast |B|$.

Solution. ≥, by the Mapping Rule.

(c) If $f$ is a surjection, then $|f(A)| \ast |B|$.

Solution. =, since $f(A) = B$.

(d) If $f$ is an injection, then $|f(A)| \ast |A|$.

Solution. =.

(e) If $f$ is a bijection, then $|A| \ast |B|$.

Solution. =, by the Mapping Rule.

Problem 4.
Let $R : A \rightarrow B$ be a binary relation. Use an arrow counting argument to prove the following generalization of the Mapping Rule 1.

Lemma. If $R$ is a function, and $X \subseteq A$, then

$$|X| \geq |R(X)|.$$ 

Solution. Proof. The proof is virtually a repeat of the arrow-counting proof in the text of Mapping Rule 1, namely:

Since $R$ is a function, at most one arrow leaves each element of $X$, so the number of arrows whose starting point is an element of $X$ is at most the number of elements in $X$, That is,

$$|X| \geq \#\text{arrows from } X.$$ 

Also, each element of $R(X)$ is, by definition, the endpoint of at least one arrow starting from $X$, so there must be at least as many arrows starting from $X$ as the number of elements of $R(X)$. That is,

$$\#\text{arrows from } X \geq |R(X)|.$$ 

Combining these inequalities immediately implies that $|X| \geq |R(X)|$.

An alternative proof appeals to the original Mapping Rule:

Proof. Let $R'$ be the relation $R$ restricted to $X$. That is, $R'$ has domain $X$, codomain $R(X)$, and the same arrows as $R$. Then $R'$ is a function because $R$ is, and $R'$ has the $[\geq 1\text{ in}]$ surjective property by definition of its codomain. Hence the surjective function Mapping Rule 1 applied to the surjective function $R' : X \rightarrow R(X)$ implies that $|X| \geq |R(X)|$.

STAFF NOTE: Here’s a repeat of the proof of Mapping Rule 1 to remind students of if need be:

Lemma (Mapping Rule). If $R : A \rightarrow B$ is a surjective function, then

$$|A| \geq |B|.$$
Proof. Since $R$ is a function, every element of $A$ contributes at most one arrow to the diagram for $R$, so the number of arrows is at most the number of elements in $A$:

$$|A| \geq \#\text{arrows}.$$  

Similarly, since $R$ is surjective, every element of $B$ has at least one arrow into it, so there must be at least as many arrows as the number of elements of $B$:

$$\#\text{arrows} \geq |B|.$$  

Combining these inequalities immediately implies that $|A| \geq |B|$.

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Problem 5. **STAFF NOTE:** Have students look up the definitions of surj and inj:

**Definition.** $A$ surj $B$ iff there is a surjective function $([\leq 1 \text{ out}, \geq 1 \text{ in}])$ from $A$ to $B$.

$A$ inj $B$ iff there is a total injective relation $([\geq 1 \text{ out}, \leq 1 \text{ in}])$ from $A$ to $B$.

For finite sets, everything below follows trivially from the Mapping Lemma about sizes of sets. Congratulate any students who get it that way, but then challenge them to do it for arbitrary sets.

The proofs below would all be clearer using an archery argument. Encourage students to do their proofs in terms of arrows-in and -out, but make sure it’s sound and clear.

(a) Prove that if $A$ surj $B$ and $B$ surj $C$, then $A$ surj $C$.

**Solution.** By definition of surj, there are surjective functions, $F : A \rightarrow B$ and $G : B \rightarrow C$.

Let $H := G \circ F$ be the function equal to the composition of $G$ and $F$, that is

$$H(a) := G(F(a)).$$

We show that $H$ is surjective, which will complete the proof. So suppose $c \in C$. Then since $G$ is a surjection, $c = G(b)$ for some $b \in B$. Likewise, $b = F(a)$ for some $a \in A$. Hence $c = G(F(a)) = H(a)$, proving that $c$ is in the range of $H$, as required.

(b) Explain why $A$ surj $B$ iff $B$ inj $A$.

**Solution.** _Proof._ (right to left): By definition of inj, there is a total injective relation, $R : B \rightarrow A$. But this implies that $R^{-1}$ is a surjective function from $A$ to $B$.

(left to right): By definition of surj, there is a surjective function, $F : A \rightarrow B$. But this implies that $F^{-1}$ is a total injective relation from $A$ to $B$.

(c) Conclude from (a) and (b) that if $A$ inj $B$ and $B$ inj $C$, then $A$ inj $C$.

**Solution.** From (b) and (a) we have that if $C$ inj $B$ and $B$ inj $A$, then $C$ inj $A$, so just switch the names $A$ and $C$.

(d) Explain why $A$ inj $B$ iff there is a total injective function $([\geq 1 \text{ out}, \leq 1 \text{ in}])$ from $A$ to $B$.

**Solution.** Given a $[\geq 1 \text{ out}, \leq 1 \text{ in}]$ relation, just erase all but one arrow wherever there is more than one arrow out of the same domain element to get an $[\leq 1 \text{ out}, \leq 1 \text{ in}]$ relation.

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\[ \text{The official definition of inj is with a total injective relation } ([\geq 1 \text{ out}, \leq 1 \text{ in}]) \]