Staff Solutions to In-Class Problems Week 1, Wed.

Problem 1.
The Pythagorean Theorem says that if $a$ and $b$ are the lengths of the sides of a right triangle, and $c$ is the length of its hypotenuse, then

$$a^2 + b^2 = c^2.$$ 

This theorem is so fundamental and familiar that we generally take it for granted. But just being familiar doesn’t justify calling it “obvious”—witness the fact that people have felt the need to devise different proofs of it for millennia.\footnote{Over a hundred different proofs are listed on the mathematics website \url{http://www.cut-the-knot.org/pythagoras/}.} In this problem we’ll examine a particularly simple “proof without words” of the theorem.

Here’s the strategy. Suppose you are given four different colored copies of a right triangle with sides of lengths $a$, $b$, and $c$, along with a suitably sized square, as shown in Figure 1.

(a) You will first arrange the square and four triangles so they form a $c \times c$ square. From this arrangement you will see that the square is $(b - a) \times (b - a)$.

**Solution.** The arrangement is given in Figure 2.

Notice that the length $a$ side of the red triangle plus the length of the square side equals the length of the $b$ side of the light gray triangle, so the square side must be of length $b - a$.

(b) You will then arrange the same shapes so they form two squares, one $a \times a$ and the other $b \times b$.

**Solution. Hint:** The two squares are attached.

The arrangement is given in Figure 3.

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\footnote{Over a hundred different proofs are listed on the mathematics website \url{http://www.cut-the-knot.org/pythagoras/}.}
You know that the area of an $s \times s$ square is $s^2$. So appealing to the principle that

Area is Preserved by Rearranging,

you can now conclude that $a^2 + b^2 = c^2$, as claimed.

This really is an elegant and convincing proof of the Pythagorean Theorem, but it has some worrisome features. One concern is that there might be something special about the shape of these particular triangles and square that makes the rearranging possible—for example, suppose $a = b$?

(c) How would you respond to this concern?

**Solution.** The justification for being able to rearrange triangles of arbitrary shape with a matching square as shown in Figures 2 and 3 uses only a few basic facts about triangles, most notably that complementary angles of a right triangle sum to a right angle. The case $a = b$ raises no problem in fitting the shapes together, except that the square reduces to a point with zero area.

(d) Another concern is that a number of facts about right triangles, squares and lines are being implicitly assumed in justifying the rearrangements into squares. Enumerate some of these assumed facts.

**STAFF NOTE:** Don’t let students blow this off. Report to instructors any additional assumptions students may identify.

**Solution.**

- Complementary angles of a right triangle sum to a right angle (used in four places in each of Figures 2 and 3).
- Lining up two right angled corners yields a straight line (also used in four places in each of Figures 2 and 3).
- Lengths along a line add up (used in four places in Figure 2 and five places in Figure 3).
- …?
Problem 2.
What’s going on here?!

\[ 1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1} \sqrt{-1} = (\sqrt{-1})^2 = -1. \]

(a) Precisely identify and explain the mistake(s) in this *bogus* proof.

**Solution.** The familiar rule that \( \sqrt{r^s} = \sqrt{r} \sqrt{s} \) (see part (c)) only holds for *positive* real numbers \( r, s \) (OK, it holds for zero too). It is not valid for negative real numbers, as the middle equality shows.

(b) Prove (correctly) that if \( 1 = -1 \), then \( 2 = 1 \).

**Solution.**

\[
\begin{align*}
1 &= -1, \\
\frac{1}{2} &= -\frac{1}{2}, \\
2 &= 1 \\
\end{align*}
\]

multiply both sides by \( \frac{1}{2} \)

add \( \frac{3}{2} \) to both sides.

If you deduce something false starting from true premises, then your reasoning must be mistaken. On the other hand, this example points that starting from a false premise, other false conclusions can be deduced by *correct* reasoning.

Indeed, basic rules of logic imply that if you assume something false, you can prove *anything*. A story illustrating this fact is told about the Nobel Prize winning logician/philosopher Bertrand Russell, who supposedly was challenged by a socialite at a party to prove from \( 1 = -1 \) that he was the Pope. Russell was famous for his quick wit, and he is said to have reasoned as above that if \( 1 = -1 \) then \( 2 = 1 \). He went on
to observe, “Now I and the Pope are clearly two, but since $2 = 1$, I and the Pope are one, that is, I am the Pope.”

(c) Every positive real number, $r$, has two square roots, one positive and the other negative. The standard convention is that the expression $\sqrt{r}$ refers to the positive square root of $r$. Assuming familiar properties of multiplication of real numbers, prove that for positive real numbers $r$ and $s$,

$$\sqrt{rs} = \sqrt{r} \sqrt{s}.$$  

**Solution.** Since $\sqrt{rs}$ refers to the positive square root of $rs$, we just have to verify that the positive number $\sqrt{r} \sqrt{s}$ is a square root of $rs$, that is, that

$$(\sqrt{r} \sqrt{s})^2 = rs.$$  

This follows directly from the fact that parentheses can be ignored in multiplications (in Math jargon, multiplication is associative) and that multiplications can be reordered (multiplication is commutative), so

$$
(\sqrt{r} \sqrt{s})^2 = (\sqrt{r} \sqrt{s})(\sqrt{r} \sqrt{s}) \quad \text{(def of squaring)} \\
= \sqrt{r} \sqrt{s} \sqrt{r} \sqrt{s} \quad \text{(ignoring parens)} \\
= \sqrt{r} \sqrt{r} \sqrt{s} \sqrt{s} \quad \text{(reordering)} \\
= (\sqrt{r})^2 (\sqrt{s})^2 \quad \text{(def of squaring)} \\
= rs \quad \text{(def of square root)}.
$$

Problem 3.
Identify exactly where the bugs are in each of the following bogus proofs.\(^2\)

(a) **Bogus Claim:** $1/8 > 1/4$.

*Bogus proof.*

$$
3 > 2 \\
3 \log_{10}(1/2) > 2 \log_{10}(1/2) \\
\log_{10}(1/2)^3 > \log_{10}(1/2)^2 \\
(1/2)^3 > (1/2)^2,
$$

and the claim now follows by the rules for multiplying fractions.

**Solution.** $\log x < 0$, for $0 < x < 1$, so since both sides of the inequality “$3 > 2$” are being multiplied by the negative quantity $\log_{10}(1/2)$, the “$>$” in the second line should have been “$<$.”

(b) **Bogus proof:** $1\epsilon = 0.01 = (0.1)^2 = (10\epsilon)^2 = 100\epsilon = 1$.

**Solution.** $0.01 = (0.1)^2 \neq (0.1)^2$ because the units $\text{S}^2$ and $\text{S}$ don’t match (just as in physics the difference between $\text{sec}^2$ and $\text{sec}$ indicates the difference between acceleration and velocity). Similarly, $(10\epsilon)^2 \neq 100\epsilon$.

\(^2\)From [42], *Twenty Years Before the Blackboard* by Michael Stueben and Diane Sandford
(c) **Bogus Claim**: If $a$ and $b$ are two equal real numbers, then $a = 0$.

**Bogus proof.**

\[
\begin{align*}
a &= b \\
 a^2 &= ab \\
 a^2 - b^2 &= ab - b^2 \\
 (a - b)(a + b) &= (a - b)b \\
 a + b &= b \\
 a &= 0.
\end{align*}
\]

**Solution.** The bug is at the fifth line: one cannot cancel $(a - b)$ from both sides of the equation on the fourth line because $a - b = 0$.

**Problem 4.**

It’s a fact that the Arithmetic Mean is at least as large as the Geometric Mean, namely,

\[
\frac{a + b}{2} \geq \sqrt{ab}
\]

for all nonnegative real numbers $a$ and $b$. But there’s something objectionable about the following proof of this fact. What’s the objection, and how would you fix it?

**Bogus proof.**

\[
\begin{align*}
\frac{a + b}{2} &\geq \sqrt{ab}, \\
 a + b &\geq 2\sqrt{ab}, \\
 a^2 + 2ab + b^2 &\geq 4ab, \\
 a^2 - 2ab + b^2 &\geq 0, \\
 (a - b)^2 &\geq 0
\end{align*}
\]

The last statement is true because $a - b$ is a real number, and the square of a real number is never negative. This proves the claim.

**Solution.** In this argument, we started with what we wanted to prove and then reasoned until we reached a statement that is surely true. The little question marks presumably are supposed to indicate that we’re not quite certain that the inequalities are valid until we get down to the last step. At that step, the inequality checks out, but that doesn’t prove the claim. All we have proved is that if $(a + b)/2 \geq \sqrt{ab}$, then $(a - b)^2 \geq 0$, which is not very interesting, since we already knew that the square of any nonnegative number is nonnegative.

To be fair, this bogus proof is pretty good: if it was written in reverse order—or if “so” was simply replaced by “is implied by” after each line—it would actually prove the Arithmetic-Geometric Mean Inequality:
Proof.

\[
\frac{a + b}{2} \geq \sqrt{ab} \quad \text{is implied by}\n\]
\[
a + b \geq 2\sqrt{ab}, \quad \text{which is implied by}\n\]
\[
a^2 + 2ab + b^2 \geq 4ab, \quad \text{which is implied by}\n\]
\[
a^2 - 2ab + b^2 \geq 0, \quad \text{which is implied by}\n\]
\[
(a - b)^2 \geq 0. \quad \text{which is implied by}\n\]

The last statement is true because \(a - b\) is a real number, and the square of a real number is never negative. This proves the claim. ■

But the problem with the bogus proof as written is that it reasons backward, beginning with the proposition in question and reasoning to a true conclusion. This kind of backward reasoning can easily “prove” false statements. Here’s an example:

**Bogus Claim:** \(0 = 1\).

**Bogus proof.**

\[
0 \overset{?}{=} 1, \quad \text{so}\n\]
\[
1 \overset{?}{=} 0, \quad \text{so}\n\]
\[
0 + 1 \overset{?}{=} 1 + 0, \quad \text{so}\n\]
\[
1 = 1 \quad \text{which is trivially true},
\]

which proves \(0 = 1\). ■

We can also come up with very easy “proofs” of true theorems, for example, here’s an easy “proof” of the Arithmetic-Geometric Mean Inequality:

**Bogus proof.**

\[
\frac{a + b}{2} \overset{?}{=} \sqrt{ab}, \quad \text{so}\n\]
\[
0 \cdot \frac{a + b}{2} \overset{?}{=} 0 \cdot \sqrt{ab}, \quad \text{so}\n\]
\[
0 \geq 0 \quad \text{which is trivially true}. ■
\]

So watch out for backward proofs!

■

**Optional (and controversial)**

**Problem 5.**

Albert announces to his class that he plans to surprise them with a quiz sometime next week.

His students first wonder if the quiz could be on Friday of next week. They reason that it can’t: if Albert didn’t give the quiz before Friday, then by midnight Thursday, they would know the quiz had to be on Friday, and so the quiz wouldn’t be a surprise any more.

Next the students wonder whether Albert could give the surprise quiz Thursday. They observe that if the quiz wasn’t given before Thursday, it would have to be given on the Thursday, since they already know it can’t be given on Friday. But having figured that out, it wouldn’t be a surprise if the quiz was on Thursday either. Similarly, the students reason that the quiz can’t be on Wednesday, Tuesday, or Monday. Namely,
it’s impossible for Albert to give a surprise quiz next week. All the students now relax, having concluded that Albert must have been bluffing. And since no one expects the quiz, that’s why, when Albert gives it on Tuesday next week, it really is a surprise!

What, if anything, do you think is wrong with the students’ reasoning?

**STAFF NOTE:** Students sometimes get caught up in elaborate answers to this problem, invoking probabilistic reasoning, for example, to explain something about “surprise.” Gently warn them off working too hard on this only to be disappointed by its unanswerability.

Other students simple deny the argument that the quiz can’t be a surprise on Thursday— for unclear reasons of course, but something like the claim that “On Thursday before class, you don’t know whether the quiz will be on Thursday or Friday, so you’ll be surprised if it happens Thursday.” It can help to go back to the argument that the quiz can’t be a surprise on Friday and ask whether or not they agree with that.

**Solution.** The basic problem is that “surprise” is not a mathematical concept, nor is there any generally accepted way to give it a mathematical definition. The “proof” above assumes some plausible axioms about surprise, without defining it. The paradox is that these axioms are inconsistent. But that’s no surprise: mathematically speaking, we don’t know what we’re talking about.

Surprise is clearly related to what you know and don’t know, and just explaining what “knowing” means is an important, but controversial and tricky problem. For example, is it possible to “know” something that isn’t true? Do you necessarily know what you know? These questions take on practical importance in Computer Science, for example in determining the limitations of distributed communication protocols [15].

Mathematicians and philosophers have had a lot more to say about what might be wrong with the students’ reasoning—see *The surprise examination or unexpected hanging paradox* by Timothy Y. Chow, [9].