Staff Solutions to In-Class Problems Week 12, Wed.

**STAFF NOTE:** Conditional Probability, Ch. 17-17.5

In Spring ’13, most teams finished 20 or more minutes early, mainly because staff was not adequately briefed to get discussions going as directed in individual problems. Also, PS_conditional_probability_problem_errors did not make it into the actual handout used in class.

**Problem 1.**

There is an unpleasant degenerative disease called Beaver Fever which causes people to tell unrelenting math jokes in social settings, believing other people would think they’re funny. Fortunately, Beaver Fever is rare, afflicting only about 1 in 1000 people. Doctor Meyer has a fairly reliable diagnostic test to determine who is going to suffer from this disease:

- If a person will suffer from Beaver Fever, the probability that Dr. Meyer diagnoses this is 0.99.
- If a person will not suffer from Beaver Fever, the probability that Dr. Meyer diagnoses this is 0.97.

Let $B$ be the event that a randomly chosen person will suffer Beaver Fever, and $Y$ be the event that Dr. Meyer’s diagnosis is “Yes, this person will suffer from Beaver Fever,” with $\overline{B}$ and $\overline{Y}$ being the complements of these events.

(a) The description above explicitly gives the values of the following quantities. What are their values?

\[
\begin{align*}
\Pr[B] & \quad \Pr[Y \mid B] \quad \Pr[\overline{Y} \mid \overline{B}] \\
0.001 & \quad 0.99 \quad 0.97
\end{align*}
\]

**Solution.** \( \Pr[B] = 0.001 \), \( \Pr[Y \mid B] = 0.99 \), \( \Pr[\overline{Y} \mid \overline{B}] = 0.97 \).

(b) Write formulas for \( \Pr[\overline{B}] \) and \( \Pr[Y \mid \overline{B}] \) solely in terms of the explicitly given quantities in part (a)—literally use their expressions, not their numeric values.

**Solution.** \( \Pr[\overline{B}] = 1 - \Pr[B] \), \( \Pr[Y \mid \overline{B}] = 1 - \Pr[\overline{Y} \mid \overline{B}] \).

(c) Write a formula for the probability that Dr. Meyer says a person will suffer from Beaver Fever solely in terms of \( \Pr[B] \), \( \Pr[\overline{B}] \), \( \Pr[Y \mid B] \) and \( \Pr[Y \mid \overline{B}] \).

**Solution.** By the Total Probability Law:

\[
\Pr[Y] = \Pr[Y \mid B] \Pr[B] + \Pr[Y \mid \overline{B}] \Pr[\overline{B}]
\]

The values turn out to be \( 0.99(1/1000) + 0.03(1 - 1/1000) = 0.03096 \).
(d) Write a formula solely in terms of the expressions given in part (a) for the probability that a person will suffer Beaver Fever given that Doctor Meyer says they will. Then calculate the numerical value of the formula.

Solution.

\[
Pr[B \mid Y] = \frac{Pr[Y \land B]}{Pr[Y]} = \frac{Pr[Y \mid B] \cdot Pr[B]}{Pr[Y \mid B] \cdot Pr[B] + Pr[Y \mid \neg B] \cdot Pr[\neg B]}
\]

The values turn out to be

\[
Pr[B \mid Y] = \frac{0.99(1/1000)}{0.03096} = \frac{99}{3096} \approx \frac{1}{32}
\]

The low probability of actually having Beaver Fever even though the (97% accurate) test says you do is because there are way more people without the disease than those with the disease. Among 1000 people, the number of false positives (999×3%) is more than 30 times the number of true positives (1×99%). So if the test says you have Beaver Fever, it’s probably a false positive.

Of course Dr. Meyer has a recourse to a 99.9% accurate test that has no false positives: simply telling everyone they don’t have the disease.

Suppose there was a vaccine to prevent Beaver Fever, but the vaccine was expensive or slightly risky itself. If you were sure you were going to suffer from Beaver Fever, getting vaccinated would be worthwhile, but by part (d), even if Dr. Meyer diagnosed you as a future sufferer of Beaver Fever, the probability you actually will suffer Beaver Fever remains low (less than 1/30).

In this case, you might sensibly decide not to be vaccinated (after all, Beaver Fever is not that bad an affliction). So the diagnostic test serves no purpose in your case—you may as well not have bothered to get diagnosed. Even so, the test may be useful:

(e) Suppose Dr. Meyer had enough vaccine to treat 2% of the population. If he randomly chose people to vaccinate, he could expect to vaccinate only 2% of the people who needed it. But by testing everyone and only vaccinating those diagnosed as future sufferers, he can expect to vaccinate a much larger fraction people who were going to suffer from Beaver Fever. Estimate this fraction.

Solution. The test will diagnose about 3% of the population as future sufferers. This 3% will include 99% of the actual sufferers but mostly include people who will not get Beaver Fever—the false positives. By giving the vaccine only to this 3% that are diagnosed as future sufferers, Dr. Meyer will have enough vaccine for 2/3 of them. So he will be able to vaccinate nearly 2/3 of the people who actually need it.

So even though the probability that a diagnosed person will suffer Beaver Fever is small, the increased probability (from 1/1000 to more than 1/32) provided by the diagnosis has significant value for public health.

Problem 2.

There are three prisoners in a maximum-security prison for fictional villains: the Evil Wizard Voldemort, the Dark Lord Sauron, and Little Bunny Foo-Foo. The parole board has declared that it will release two of
the three, chosen uniformly at random, but has not yet released their names. Naturally, Sauron figures that he will be released to his home in Mordor, where the shadows lie, with probability $\frac{2}{3}$.

A guard offers to tell Sauron the name of one of the other prisoners who will be released (either Voldemort or Foo-Foo). If the guard has a choice of naming either Voldemort or Foo-Foo (because both are to be released), he names one of the two with equal probability.

Sauron knows the guard to be a truthful fellow. However, Sauron declines this offer. He reasons that knowing what the guards says will reduce his chances, so he is better off not knowing. For example, if the guard says, “Little Bunny Foo-Foo will be released”, then his own probability of release will drop to $\frac{1}{2}$ because he will then know that either he or Voldemort will also be released, and these two events are equally likely.

Dark Lord Sauron has made a typical mistake when reasoning about conditional probability. Using a tree diagram and the four-step method, explain his mistake. What is the probability that Sauron is released given that the guard says Foo-Foo is released?

**Hint:** Define the events $S$, $F$, and “$F$” as follows:

- “$F$” = Guard says Foo-Foo is released
- $F$ = Foo-Foo is released
- $S$ = Sauron is released

**Solution.** Sauron’s mistake can be explained as his confusing the two different events $F$ and “$F$”. His observation that $\Pr [S \mid F] = \frac{1}{2}$ is correct, but that’s the wrong thing to calculate. He should be calculating $\Pr [S \mid “F”]$.

To clarify the error and work out the proper probability, let’s begin by working out the sample space, noting events of interest, and computing outcome probabilities:

The outcomes in each of these events are noted in the tree diagram.

The tree shows how the event $F$ (Foo-foo will be released) is different from the event “$F$” (the guard says Foo-foo will be released). In particular, the probability that Sauron is released, given that Foo-foo is released, is indeed $\frac{1}{2}$:

$$
\Pr [S \mid F] = \frac{\Pr [S \cap F]}{\Pr [F]}
$$

$$
= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6} + \frac{1}{6}}
$$

$$
= \frac{1}{2}
$$
But the probability that Sauron is released given that the guard actually says so is still \( \frac{2}{3} \):

\[
\Pr[S \mid "F"] = \frac{\Pr[S \cap "F"]}{\Pr["F"]} = \frac{1/3}{1/3 + 1/6} = \frac{2}{3}
\]

So Sauron’s probability of release is unchanged by the guard’s statement.

Problem 3.

There are two decks of cards. One is complete, but the other is missing the Ace of spades. Suppose you pick one of the two decks with equal probability and then select a card from that deck uniformly at random. What is the probability that you picked the complete deck, given that you selected the eight of hearts? Use the four-step method and a tree diagram.

**STAFF NOTE:** Try to get a brief discussion going on the issue “How could knowing about the eight of hearts be relevant to the presence of the Ace of spades?”

**Solution.** Let \( C \) be the event that you pick the complete deck, and let \( H \) be the event that you select the eight of hearts. In these terms, our aim is to compute:

\[
\Pr[C \mid H] = \frac{\Pr[C \cap H]}{\Pr[H]}
\]

A tree diagram is worked out below:

Now we can compute the desired conditional probability as follows:
Thus, if you selected the eight of hearts, then the deck you picked is less likely to be the complete one. It’s worth stopping to think about how you might have arrived at this final conclusion without going through the detailed calculation—or better, how you might explain it to your 10-year-old niece.

The explanation is simple: drawing an eight of hearts from a small deck containing an eight of hearts is more likely than drawing one from a larger such deck. So if you see an eight of hearts, it’s more likely to have come from a smaller deck. The soundness of this intuitive explanation is proved in Problem 17.16.

Supplemental problem

Problem 4.
Suppose you repeatedly flip a fair coin until you see the sequence HHT or HHT. What is the probability you see the sequence HHT first?

Hint: Try to find the probability that HHT comes before HHT conditioning on whether you first toss an H or a T. The answer is not 1/2.

Solution. Let \( A \) be the event that HHT appears before HHT, and let \( p := \Pr[A] \).

Suppose our first toss is T. Since neither of our patterns starts with T, the probability that \( A \) will occur from this point on is still \( p \). That is, \( \Pr[A | T] = p \).

Suppose our first toss is H. To find the probability that \( A \) will now occur, that is, to find \( r := \Pr[A | H] \), we consider different cases based on the subsequent throws.

Suppose the next toss is H, that is, the first two tosses are HH. Then neither pattern appears if we continue flipping H, and when we eventually toss a T, the pattern HHT will then have appeared first. So in this case, event \( A \) will never occur. That is \( \Pr[A | HH] = 0 \).

Suppose the first two tosses are HT. If we toss a T again, then we have tossed HHT, so event \( A \) has occurred. If we next toss an H, then we have tossed HTH. But this puts us in the same situation we were in after rolling an H on the first toss. That is, \( \Pr[A | HTH] = r \).

Summarizing this we have:

\[
\Pr[A] = \Pr[A | T] \Pr[T] + \Pr[A | H] \Pr[H] \quad \text{(Law of Total Probability)}
\]

\[
p = p \frac{1}{2} + r \frac{1}{2}
\]

so

\[
p = r.
\]
Continuing, we have
\[
\Pr[A \mid H] = \Pr[A \mid HT] \Pr[T] + \Pr[A \mid HH] \Pr[H] \quad \text{(Law of Total Probability)}
\]
\[
r = \Pr[A \mid HT] \frac{1}{2} + 0 \cdot \frac{1}{2}
\]
\[
\Pr[A \mid HT] = \Pr[A \mid HTT] \Pr[T] + \Pr[A \mid HTH] \Pr[H] \quad \text{(Law of Total Probability)}
\]
\[
r = \frac{1}{2} + r \frac{1}{2}
\]
\[
r = \frac{1}{3} \quad \text{by (1) & (2)}
\]

So HTT comes before HHT with probability
\[
p = r = \frac{1}{3}.
\]

These kind of events have an amazing intransitivity property: if you pick any pattern of three tosses such as HTT, then I can pick a pattern of three tosses such as HHT whose odds of coming up first are better than even. If we then bet on which pattern will appear first in a series of tosses, the odds will be in my favor. In particular, even if you instead picked the “better” pattern HHT, there is another pattern I can pick that has a more than even chance of appearing before HHT. So there are cases where some pattern 1 appears before pattern 2 with probability better than one half, and pattern 2 appears before some pattern 3 with probability better than one half, but pattern 1 does not appear before pattern 3 with probability better than one half.

Watch out for this intransitivity phenomenon if somebody proposes that you bet real money on coin flips. □