Staff Solutions to In-Class Problems Week 12, Fri.

STAFF NOTE: Independence, Ch. 17.7-17.8

Problem 1.
Sally Smart just graduated from high school. She was accepted to three reputable colleges.

- With probability $4/12$, she attends Yale.
- With probability $5/12$, she attends MIT.
- With probability $3/12$, she attends Little Hoop Community College.

Sally is either happy or unhappy in college.

- If she attends Yale, she is happy with probability $4/12$.
- If she attends MIT, she is happy with probability $7/12$.
- If she attends Little Hoop, she is happy with probability $11/12$.

(a) A tree diagram to help Sally project her chance at happiness is shown below. On the diagram, fill in the edge probabilities, and at each leaf write the probability of the corresponding outcome.

Solution. See Figure 1.

(b) What is the probability that Sally is happy in college?

Figure 1  Probability tree for College happiness

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Solution. The probability that Sally is happy is equal to the sum of the probabilities of the outcomes marked with an “H”: 16/144 + 35/144 + 33/144 = 84/144 = 7/12.

(c) What is the probability that Sally attends Yale, given that she is happy in college?

Solution.

\[
\Pr[\text{attends Yale} \mid \text{happy}] = \frac{\Pr[\text{attends Yale} \cap \text{happy}]}{\Pr[\text{happy}]} = \frac{(4/12) \cdot (4/12)}{(7/12)} = \frac{4}{21}
\]

(d) Show that the event that Sally attends Yale is not independent of the event that she is happy.

Solution. These two events are independent only if

\[
\Pr[\text{attends Yale} \mid \text{happy}] = \Pr[\text{attends Yale}]
\]

or \(\Pr[\text{happy}] = 0\). However, the left side is 4/21, the right side is 4/12, and the probability that Sally is happy is nonzero.

(e) Show that the event that Sally attends MIT is independent of the event that she is happy.

Solution. These two events are independent if

\[
\Pr[\text{attends MIT}] \cdot \Pr[\text{happy}] = \Pr[\text{attends MIT} \cap \text{happy}]
\]

The left side is equal to (5/12) \cdot (7/12) = 35/144. According to the tree diagram, the right side is equal to 35/144 as well.

Problem 2.
Suppose you flip three fair, mutually independent coins. Define the following events:

- Let \(A\) be the event that the first coin is heads.
- Let \(B\) be the event that the second coin is heads.
- Let \(C\) be the event that the third coin is heads.
- Let \(D\) be the event that an even number of coins are heads.

(a) Use the four step method to determine the probability space for this experiment and the probability of each of \(A, B, C, D\).

Solution. The tree is a binary tree with depth 3 and 8 leaves. The successive levels branch to show whether or not the successive events \(A, B, C\) occur. By the definitions of the characteristics fair and independent, each branch from a vertex is equally likely to be followed. So the probability space has, as outcomes, eight length-3 strings of \(H\)'s and \(T\)'s, each of which has probability \((1/2)^3 = 1/8\).

Each of the events \(A, B, C, D\) are true in four of the outcomes and hence has probability 1/2.
We conclude by symmetry that $B$ and $D$ are also independent. The above completes the verification that $A, B, C, D$ are 3-way independent.

Problem 3.

Graphs, Logic & Probability

Let $G$ be an undirected simple graph with $n > 3$ vertices. Let $E(x, y)$ mean that $G$ has an edge between vertices $x$ and $y$, and let $P(x, y)$ mean that there is a length 2 walk in $G$ between $x$ and $y$.

(a) Write a predicate-logic formula defining $P(x, y)$ in terms of $E(x, y)$.

Solution. $P(x, y) := \exists z, z \neq x \text{ AND } z \neq y \text{ AND } E(x, z) \text{ AND } E(z, y)$

For the following parts (b) – (d), let $V$ be a fixed set of $n > 3$ vertices, and let $G$ be a graph with these vertices constructed randomly as follows: for all distinct vertices $x, y \in V$, independently include edge $\langle x — y \rangle$ as an edge of $G$ with probability $p$. In particular, $\Pr[E(x, y)] = p$ for all $x \neq y$.

(b) For distinct vertices $w, x, y$ and $z$ in $V$, circle the event pairs that are independent.

STAFF NOTE: In class, ask for explanations.

Here our sample space consists of sets of edges. In answering each individual question below, we rely on a key property: since edges are selected independently, two events are independent if they depend on disjoint sets of edges. Formally, we say that an event $E$ depends on edge $\langle u — v \rangle$ iff there exists $\omega$ such that exactly one of $\omega$ and $\omega \cup \{\langle u — v \rangle\}$ is in $E$.

1. $E(w, x)$ versus $E(x, y)$ TRUE, as obviously each side depends on a single, different edge.
2. \([E(w, x) \text{ AND } E(w, y)] \text{ versus } [E(z, x) \text{ AND } E(z, y)]\) \text{ TRUE}, as obviously each side depends on two edges, and no edge could appear on both sides.

3. \(E(x, y) \text{ versus } P(x, y)\) \text{ TRUE}, as \((x-y)\) can’t possibly be involved in a length-2 walk from \(x\) to \(y\), since it could only be connected to a self-loop for \(x\) or \(y\) to itself, which we disallow in simple graphs.

4. \(P(w, x) \text{ versus } P(x, y)\) \text{ FALSE}, as demonstrated by the counterexample of \(|V| = 4\) and \(p = \frac{1}{2}\), where \(P(w, x) \equiv (E(w, y) \text{ AND } E(y, x)) \text{ OR } (E(w, z) \text{ AND } E(z, x))\) and \(P(x, y) \equiv (E(x, w) \text{ AND } E(w, y)) \text{ OR } (E(x, z) \text{ AND } E(z, y))\). By symmetry, we apply inclusion-exclusion to calculate the probability for either of these events: \((\frac{1}{2})^2 + (\frac{1}{2})^2 - (\frac{1}{2})^4 = \frac{7}{16}\). Now consider \(\Pr[P(w, x) \mid P(x, y)\] or \(P(w, x)\) that also satisfy \(P(w, x)\). Partition the outcomes satisfying \(P(x, y)\) by whether they also satisfy \(E(w, y)\) and \(E(w, y)\). Both sides of the partition are independent of \(E(w, y)\) in the sense formalized above, since \(E(y, x)\) doesn’t appear in the definition of \(P(x, y)\). That means the outcomes in the subcase for \(E(w, y)\) can be partitioned into equally sized sets, one with \(E(w, y)\) and the other with \(-E(w, y)\). Clearly every element of the first set satisfies \(P(w, x)\), so \(\Pr[P(w, x) \mid P(x, y) \text{ AND } E(w, y)] \geq \frac{1}{2}\). The outcomes in the subcase for \(-E(w, y)\) must all have \(E(w, z)\), so, like above partitioning them based on \(E(w, z)\), we get two equal-size sets, where the set with \(E(w, z)\) all satisfy \(P(w, x)\), and \(\Pr[P(w, x) \mid P(x, y) \text{ AND } -E(w, y)] \geq \frac{1}{2}\). The true value of \(\Pr[P(w, x) \mid P(x, y)]\) must lie somewhere between these two values, so it also must be no less than \(\frac{7}{16}\), and thus it must be greater than \(\Pr[P(w, x)] = \frac{7}{16}\).

5. \(P(w, x) \text{ versus } P(y, z)\) \text{ FALSE}, by similar reasoning to in the last part. For \(|V| = 4\) and \(p = \frac{1}{2}\), we have \(P(w, x) \equiv (E(w, y) \text{ AND } E(y, x)) \text{ OR } (E(w, z) \text{ AND } E(z, x))\) and \(P(y, z) \equiv (E(y, x) \text{ AND } E(x, z)) \text{ OR } (E(y, w) \text{ AND } E(w, z))\). \(\Pr[P(w, x) \mid P(x, y) \text{ AND } -E(y, x) \text{ AND } E(z, x)] = \frac{3}{4}\), since under these conditions the formula for \(P(w, x)\) simplifies to \(E(w, y)\) or \(E(w, z)\).

Since under these conditions we know \(E(y, w)\) and \(E(w, z)\), and the formula for \(P(w, x)\) simplifies to \((E(y, x) \text{ OR } E(z, x))\) \text{ AND } \(-E(y, x) \text{ AND } E(z, x))\). Both sides of the partition have probabilities no less than \(\frac{1}{2}\), so \(\Pr[P(w, x) \mid P(y, z)] \geq \frac{1}{2}\), which again is above \(\Pr[P(w, x)]\), which can be computed as \(\frac{7}{16}\) as in the prior case.

(e) Write a simple formula in terms of \(n\) and \(p\) for \(\Pr[\text{NOT } P(x, y)]\), for distinct vertices \(x\) and \(y\) in \(V\).

\text{Hint: Use part (b), item 2.}

\textbf{Solution.} Let \(Z ::= V \setminus \{x, y\}\) be the set of all the vertices other than \(x\) and \(y\).

\[
\Pr[\text{NOT } P(x, y)] = \Pr[\text{AND}_{z \in Z} E(x, z) \text{ AND } E(y, z)]
\]
\[
= \prod_{z \in Z} \Pr[E(x, z) \text{ AND } E(y, z)] \quad \text{(indep. from item (b) 2.)}
\]
\[
= \prod_{z \in Z} (1 - \Pr[E(x, z)] \cdot \Pr[E(y, z)]) \quad \text{(indep. from item (b) 1.)}
\]
\[
= \prod_{z \in Z} (1 - p^2)
\]
\[
= (1 - p^2)^{n-2}
\]
(d) What is the probability that two distinct vertices \(x\) and \(y\) lie on a three-cycle in \(G\)? Answer with a simple expression in terms of \(p\) and \(r\), where \(r := \Pr[\text{NOT}(P(x, y))]\) is the correct answer to part (c).

*Hint:* Express \(x\) and \(y\) being on a three-cycle as a simple formula involving \(E(x, y)\) and \(P(x, y)\).

**Solution.** \(x\) and \(y\) lie on a three-cycle iff \(E(x, y)\) AND \(P(x, y)\).

Since \(E(x, y)\) and \(P(x, y)\) are independent,

\[
\Pr[E(x, y) \text{ AND } P(x, y)] = \Pr[E(x, y)] \cdot \Pr[P(x, y)] = p(1 - r).
\]

Substituting in for \(r\) (not asked), we get

\[
\Pr[E(x, y) \text{ AND } P(x, y)] = p(1 - (1 - p^2)^{n-2}).
\]

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**Supplemental Problem**

**Problem 4.**

Let \(A, B, C\) be events. For each of the following statements, prove it or give a counterexample.

(a) If \(A\) is independent of \(B\), then \(A\) is also independent of \(\overline{B}\).

**Solution.** True.

To prove it, suppose \(A\) is independent of \(B\); that is, \(\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]\). Then

\[
\Pr[A \cap \overline{B}] = \Pr[A] - \Pr[A \cap B] = \Pr[A] - \Pr[A] \cdot \Pr[B] = \Pr[A](1 - \Pr[B]) = \Pr[A] \cdot \Pr[\overline{B}].
\]

(b) If \(A\) is independent of \(B\), and \(A\) is independent of \(C\), then \(A\) is independent of \(B \cap C\).

*Hint:* Choose \(A, B, C\) pairwise but not 3-way independent.

**Solution.** False.

Let \(A, B, C\) be any set of pairwise but not 3-way independent events.

So \(A\) is independent of \(B\) and of \(C\) by pairwise independence, but

\[
\begin{align*}
\Pr\left[ A \mid B \cap C \right] &:= \frac{\Pr[A \cap (B \cap C)]}{\Pr[B \cap C]} \\
&= \frac{\Pr[A] \cdot \Pr[B] \cdot \Pr[C]}{\Pr[B \cap C]} \\
&\neq \frac{\Pr[A] \cdot \Pr[B] \cdot \Pr[C]}{\Pr[B \cap C]} \quad (A, B, C \text{ not 3-way independent}) \\
&= \frac{\Pr[A] \cdot \Pr[B] \cdot \Pr[C]}{\Pr[B] \cdot \Pr[C]} \quad (B, C \text{ pairwise independent}) \\
&= \Pr[A],
\end{align*}
\]

so \(A\) is not independent of \(B \cap C\).
(c) If \( A \) is independent of \( B \), and \( A \) is independent of \( C \), then \( A \) is independent of \( B \cup C \).

*Hint:* Part (b).

**Solution. False.**

As in part (b), let \( A, B, C \) be any set of pairwise but not 3-way independent events. So \( A \) is independent of both \( B \) and \( C \) by pairwise independence.

Now by part (a), \( A \) is independent of \( B \cup C \) iff it is independent of \( \overline{B} \cup \overline{C} \). That is, \( A \) must be independent of \( \overline{B} \cap \overline{C} \).

But by part (a) again, \( A, \overline{B}, \) and \( \overline{C} \) are also pairwise but not 3-way independent, and we saw in part (b) that in this case, \( A \) is not independent of \( \overline{B} \cap \overline{C} \).

It is also easy to give a concrete counterexample. Let the probability space be the integer interval \([1, 4]\) with uniform probability, and let

\[
A := \text{[Even]}, \quad B := \lfloor 2 \rfloor, \quad C := \{2, 3\}.
\]

Then

\[
\Pr[A] = \frac{1}{2} = \Pr[A | B] = \Pr[A | C],
\]

so \( A \) is independent of \( B \) and of \( C \).

But

\[
\Pr[A | B \cup C] = \Pr[\text{[even]} | [1, 3]] = \frac{1}{3} \neq \frac{1}{2} = \Pr[A].
\]

so \( A \) is not independent of \( B \cup C \).

**Solution. True.**

To prove it, suppose \( A \) is independent of each of \( B, C, \) and \( B \cap C \), so

\[
\Pr[A \cap B] = \Pr[A] \Pr[B] \quad \text{(1)}
\]
\[
\Pr[A \cap C] = \Pr[A] \Pr[C] \quad \text{(2)}
\]
\[
\Pr[A \cap (B \cap C)] = \Pr[A] \Pr[B \cap C]. \quad \text{(3)}
\]

Then,

\[
\Pr[A \cap (B \cup C)] = \Pr[(A \cap B) \cup (A \cap C)]
\]
\[
= \Pr[A \cap B] + \Pr[A \cap C] - \Pr[(A \cap B) \cap (A \cap C)] \quad \text{(inclusion-exclusion)}
\]
\[
= \Pr[A \cap B] + \Pr[A \cap C] - \Pr[A \cap B \cap C]
\]
\[
= \Pr[A] \Pr[B] + \Pr[A] \Pr[C] - \Pr[A] \Pr[B \cap C] \quad \text{(by (1), (2), (3))}
\]
\[
= \Pr[A](\Pr[B] + \Pr[C] - \Pr[B \cap C])
\]
\[
= \Pr[A] \Pr[B \cup C]. \quad \text{(inclusion-exclusion)}
\]