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Recursive Functions



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recursivefunctions.1

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Recursive Function

To define a function, f , on a recursively defined set R , define

- $f(b)$ explicitly for each base case $b \in R$
- $f(c(x))$ for each constructor, c , in terms of x and $f(x)$



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Recursive function on M

Def. tree-depth(s) for $s \in M$

$$td(\lambda) ::= 0$$

$$td([s]t) ::=$$

$$1 + \max\{td(s), td(t)\}$$



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k^n — recursive function on \mathbb{N}

$$\text{expt}(k, 0) ::= 1$$

$$\text{expt}(k, n+1) ::= k \cdot \text{expt}(k, n)$$

--uses recursive def of \mathbb{N} :

- $0 \in \mathbb{N}$
- if $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$



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Recursive Functions

summary:

$f: \text{Data} \rightarrow \text{Values}$

$f(b)$ def'd directly for base b

$f(\text{cnstr}(x))$ def'd using $f(x), x$



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Length versus Depth

Lemma: $|r| + 2 \leq 2^{\text{td}(r)+1}$
for all $r \in M$

Proof by Structural Induction

Base case: $[r = \lambda]$

$$|\lambda| + 2 = 0 + 2 = 2 = 2^{0+1} = 2^{\text{td}(\lambda)+1}$$

OK!



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Size versus Depth

Constructor case: $[r = [s]t]$

by ind. hypothesis:

$$|s| + 2 \leq 2^{\text{td}(s)+1}$$

$$|t| + 2 \leq 2^{\text{td}(t)+1}$$



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Size versus Depth

$$\begin{aligned} |r| + 2 &= |[s]t| + 2 && \text{def. of } r \\ &= (|s| + |t| + 2) + 2 && \text{def. of length} \\ &= (|s| + 2) + (|t| + 2) \\ &\leq 2^{\text{td}(s)+1} + 2^{\text{td}(t)+1} && \text{induction hyp.} \\ &\leq 2^{\max(\text{td}(s), \text{td}(t))+1} + 2^{\max(\text{td}(s), \text{td}(t))+1} \\ &= 2 \cdot 2^{\max(\text{td}(s), \text{td}(t))+1} \leq 2 \cdot 2^{\text{td}(r)} && \text{def. of } d(r) \\ &= 2^{\text{td}(r)+1} \quad \text{QED!} \end{aligned}$$



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positive powers of two

$2 \in \text{PP2}$

if $x, y \in \text{PP2}$, then $x \cdot y \in \text{PP2}$

$2, 2 \cdot 2, 4 \cdot 2, 4 \cdot 4, 4 \cdot 8, \dots$

$2 \quad 4 \quad 8 \quad 16 \quad 32 \dots \in \text{PP2}$



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\log_2 of PP2

$$\log_2(2) ::= 1$$

$$\log_2(x \cdot y) ::= \log_2(x) + \log_2(y)$$

for $x, y \in \text{PP2}$

$$\log_2(4) = \log_2(2 \cdot 2) = 1 + 1 = 2$$

$$\begin{aligned} \log_2(8) &= \log_2(2 \cdot 4) = \log_2(2) + \log_2(4) \\ &= 1 + 2 = 3 \end{aligned}$$



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loggy function on PP2

$$\text{loggy}(2) ::= 1$$

$$\text{loggy}(x \cdot y) ::= x + \text{loggy}(y)$$

for $x, y \in \text{PP2}$

$$\text{loggy}(4) = \text{loggy}(2 \cdot 2) = 2 + 1 = 3$$

$$\begin{aligned} \text{loggy}(8) &= \text{loggy}(2 \cdot 4) = 2 + \text{loggy}(4) \\ &= 2 + 3 = 5 \end{aligned}$$

$$\begin{aligned} \text{loggy}(16) &= \text{loggy}(8 \cdot 2) = 8 + \text{loggy}(2) \\ &= 8 + 1 = 9 \end{aligned}$$



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loggy function on PP2

$$\text{loggy}(16) = \text{loggy}(8 \cdot 2) = 9$$

WAIT A SEC!

$$\begin{aligned} \text{loggy}(16) &= \text{loggy}(2 \cdot 8) \\ &= 2 + \text{loggy}(8) = 2 + 5 \\ &= 7 \end{aligned}$$



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ambiguous constructors

The Problem: more than one way to construct elements of PP2 from
 $\text{cnstrct}(x,y) = x \cdot y$

$16 = \text{cnstrct}(8,2)$ but also

$16 = \text{cnstrct}(2,8)$

ambiguous



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ambiguous recursive defs

problem to watch out for:
 recursive function on datum, e ,
 is defined according to what
 constructor created e .

If 2 or more ways to construct e ,
 then which definition to use?



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recursivefunctions.29