Problem Set 8

Due: April 10

Reading: Sections 11.7– 11.10, 11.6

Problem 1.
Prove Corollary 11.10.12: If all edges in a finite weighted graph have distinct weights, then the graph has a *unique* MST.

Hint: Suppose M and N were different MST's of the same graph. Let e be the smallest edge in one and not the other, say $e \in M - N$, and observe that $N + e$ must have a cycle.

Problem 2.
A basic example of a simple graph with chromatic number n is the complete graph on n vertices, that is $\chi(K_n) = n$. This implies that any graph with K_n as a subgraph must have chromatic number at least n. It's a common misconception to think that, conversely, graphs with high chromatic number must contain a large complete subgraph. In this problem we exhibit a simple example countering this misconception, namely a graph with chromatic number four that contains no *triangle*—length three cycle—and hence no subgraph isomorphic to K_n for $n \geq 3$. Namely, let G be the 11-vertex graph of Figure 1. The reader can verify that G is triangle-free.

(a) Show that G is 4-colorable.

(b) Prove that G can't be colored with 3 colors.

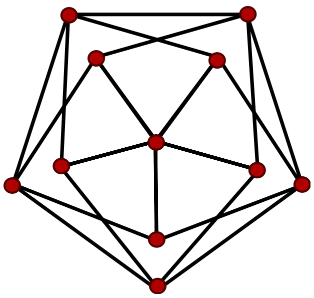

Figure 1 Graph G with no triangles and $\chi(G) = 4$.

Problem 3.
The preferences among 4 boys and 4 girls are partially specified in the following table:

B1:	G1	G2	–	–
B2:	G2	G1	–	–
B3:	–	–	G4	G3
B4:	–	–	G3	G4
G1:	B2	B1	–	–
G2:	B1	B2	–	–
G3:	–	–	B3	B4
G4:	–	–	B4	B3

(a) Verify that

$$(B1, G1), (B2, G2), (B3, G3), (B4, G4)$$

will be a stable matching whatever the unspecified preferences may be.

(b) Explain why the stable matching above is neither boy-optimal nor boy-pessimal and so will not be an outcome of the Mating Ritual.

(c) Describe how to define a set of marriage preferences among n boys and n girls which have at least $2^{n/2}$ stable assignments.

Hint: Arrange the boys into a list of $n/2$ pairs, and likewise arrange the girls into a list of $n/2$ pairs of girls. Choose preferences so that the kth pair of boys ranks the kth pair of girls just below the previous pairs of girls, and likewise for the kth pair of girls. Within the kth pairs, make sure each boy's first choice girl in the pair prefers the other boy in the pair.