Problem Set 7

Due: April 3

Reading:

- Chapter 9. *Directed Graphs* 9.5 through 9.6, and 9.8 through 9.11 (omit 9.7).
- Omit Chapter 10.
- Chapter 11. *Simple Graphs* through 11.4 (omit 11.5)

Problem 1.
Let R and S be transitive binary relations on the same set, A. Which of the following new relations must also be transitive? For each part, justify your answer with a brief argument if the new relation is transitive and a counterexample if it is not.

(a) R^{-1}

(b) $R \cap S$

(c) $R \circ R$

(d) $R \circ S$

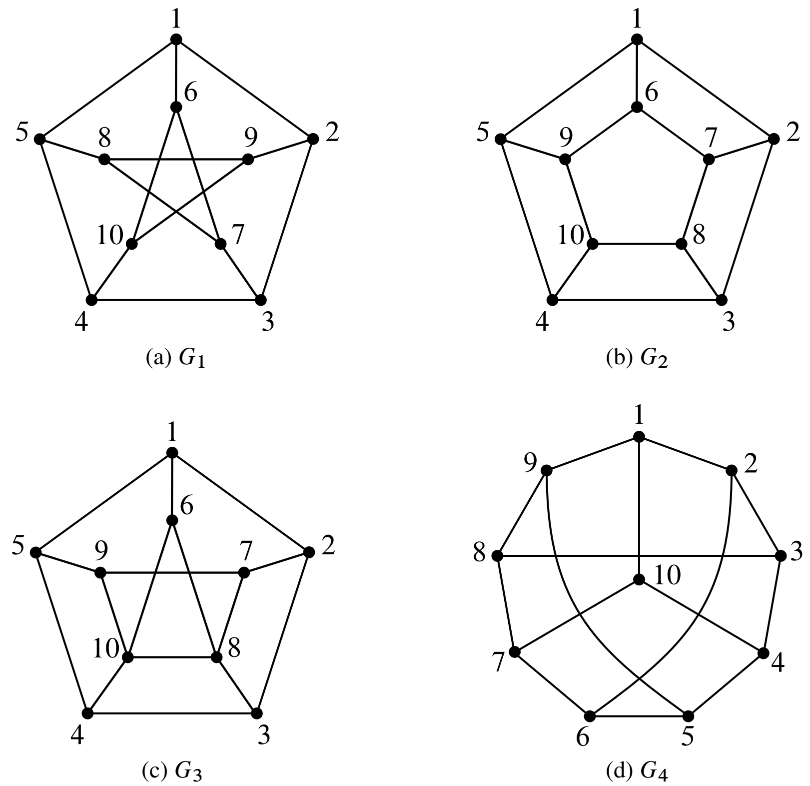
Problem 2.
Let R_1 and R_2 be two equivalence relations on a set, A. Prove or give a counterexample to the claims that the following are also equivalence relations:

(a) $R_1 \cap R_2$.

(b) $R_1 \cup R_2$.

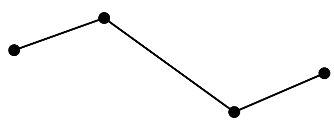
Problem 3.
Determine which among the four graphs pictured in Figure 1 are isomorphic. For each pair of isomorphic graphs, describe an isomorphism between them. For each pair of graphs that are not isomorphic, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, *prove* that it is indeed preserved under isomorphism (you only need prove one of them).

Problem 4.
Let's say that a graph has "two ends" if it has exactly two vertices of degree 1 and all its other vertices have degree 2. For example, here is one such graph:



(a) G_1 (b) G_2 (c) G_3 (d) G_4

Figure 1 Which graphs are isomorphic?



(a) A *line graph* is a graph whose vertices can be listed in a sequence with edges between consecutive vertices only. So the two-ended graph above is also a line graph of length 4.

Prove that the following theorem is false by drawing a counterexample.

False Theorem. *Every two-ended graph is a line graph.*

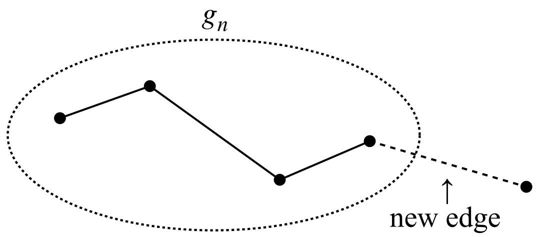
(b) Point out the first erroneous statement in the following bogus proof of the false theorem and describe the error.

Bogus proof. We use induction. The induction hypothesis is that every two-ended graph with n edges is a path.

Base case ($n = 1$): The only two-ended graph with a single edge consists of two vertices joined by an edge:

Sure enough, this is a line graph.

Inductive case: We assume that the induction hypothesis holds for some $n \geq 1$ and prove that it holds for $n + 1$. Let G_n be any two-ended graph with n edges. By the induction assumption, G_n is a line graph. Now suppose that we create a two-ended graph G_{n+1} by adding one more edge to G_n. This can be done in



only one way: the new edge must join an endpoint of G_n to a new vertex; otherwise, G_{n+1} would not be two-ended.

Clearly, G_{n+1} is also a line graph. Therefore, the induction hypothesis holds for all graphs with $n + 1$ edges, which completes the proof by induction.

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