Problem Set 4

Due: March 6

Reading:

- Section 5.4. *State Machines: Invariants*
- Chapter 6. *Recursive Data Types*
- Chapter 7. *Infinite Sets, The Halting Problem.*

Problem 1.
A robot moves on the two-dimensional integer grid. It starts out at $(0, 0)$ and is allowed to move in any of these four ways:

1. $(+2, -1)$: right 2, down 1
2. $(-2, +1)$: left 2, up 1
3. $(+1, +3)$
4. $(-1, -3)$

Prove that this robot can never reach $(1, 1)$.

Problem 2.
Let L be some convenient set whose elements will be called *labels*. The labeled binary trees, LBT's, are defined recursively as follows:

Definition. Base case: if l is a label, then $\langle l, \texttt{leaf} \rangle$ is an LBT, and

Constructor case: if B and C are LBT's, then $\langle l, B, C \rangle$ is an LBT.

The *leaf-labels* and *internal-labels* of an LBT are defined recursively in the obvious way:

Definition. Base case: The set of leaf-labels of the LBT $\langle l, \texttt{leaf} \rangle$ is $\{l\}$, and its set of internal-labels is the empty set.

Constructor case: The set of leaf labels of the LBT $\langle l, B, C \rangle$ is the union of the leaf-labels of B and of C; the set of internal-labels is the union of $\{l\}$ and the sets of internal-labels of B and of C.

The set of *labels* of an LBT is the union of its leaf- and internal-labels.
The LBT's with *unique* labels are also defined recursively:

Definition. Base case: The LBT $\langle l, \texttt{leaf}\rangle$ has *unique labels*.

Constructor case: If B and C are LBT's with unique labels, no label of B is a label C and vice-versa, and l is not a label of B or C, then $\langle l, B, C\rangle$ has *unique labels*.

If B is an LBT, let n_B be the number of distinct i**n**ternal-labels appearing in B and f_B be the number of distinct lea**f** labels of B. Prove by structural induction that

$$f_B = n_B + 1 \tag{1}$$

for all LBT's B with unique labels. This equation can obviously fail if labels are not unique, so your proof had better use uniqueness of labels at some point; be sure to indicate where.

Problem 3.
In this problem you will prove a fact that may surprise you—or make you even more convinced that set theory is nonsense: the half-open unit interval is actually the "*same size*" as the nonnegative quadrant of the real plane![1] Namely, there is a bijection from $(0, 1]$ to $[0, \infty) \times [0, \infty)$.

(a) Describe a bijection from $(0, 1]$ to $[0, \infty)$.

Hint: $1/x$ almost works.

(b) An infinite sequence of the decimal digits $\{\texttt{0}, \texttt{1}, \ldots, \texttt{9}\}$ will be called *long* if it does not end with all 0's. An equivalent way to say this is that a long sequence is one that has infinitely many occurrences of nonzero digits. Let L be the set of all such long sequences. Describe a bijection from L to the half-open real interval $(0, 1]$.

Hint: Put a decimal point at the beginning of the sequence.

(c) Describe a surjective function from L to L^2 that involves alternating digits from two long sequences. *Hint:* The surjection need not be total.

(d) Prove the following lemma and use it to conclude that there is a bijection from L^2 to $(0, 1]^2$.
Lemma 3.1. *Let A and B be nonempty sets. If there is a bijection from A to B, then there is also a bijection from $A \times A$ to $B \times B$.*

(e) Conclude from the previous parts that there is a surjection from $(0, 1]$ to $(0, 1]^2$. Then appeal to the Schröder-Bernstein Theorem to show that there is actually a bijection from $(0, 1]$ to $(0, 1]^2$.

(f) Complete the proof that there is a bijection from $(0, 1]$ to $[0, \infty)^2$.

[1] The half-open unit interval, $(0, 1]$, is $\{r \in \mathbb{R} \mid 0 < r \leq 1\}$. Similarly, $[0, \infty) ::= \{r \in \mathbb{R} \mid r \geq 0\}$.