Problem Set 4

Due: March 6

Reading:

- Section 5.4, State Machines: Invariants
- Chapter 6, Recursive Data Types
- Chapter 7, Infinite Sets, The Halting Problem.

Problem 1.
A robot moves on the two-dimensional integer grid. It starts out at (0, 0) and is allowed to move in any of these four ways:

1. (+2, −1): right 2, down 1
2. (−2, +1): left 2, up 1
3. (+1, +3)
4. (−1, −3)

Prove that this robot can never reach (1, 1).

Problem 2.
Let \( L \) be some convenient set whose elements will be called labels. The labeled binary trees, LBT’s, are defined recursively as follows:

**Definition. Base case:** if \( l \) is a label, then \((l, \text{leaf})\) is an LBT, and

**Constructor case:** if \( B \) and \( C \) are LBT’s, then \((l, B, C)\) is an LBT.

The leaf-labels and internal-labels of an LBT are defined recursively in the obvious way:

**Definition. Base case:** The set of leaf-labels of the LBT \((l, \text{leaf})\) is \(\{l\}\), and its set of internal-labels is the empty set.

**Constructor case:** The set of leaf labels of the LBT \((l, B, C)\) is the union of the leaf-labels of \( B \) and of \( C \); the set of internal-labels is the union of \(\{l\}\) and the sets of internal-labels of \( B \) and of \( C \).

The set of labels of an LBT is the union of its leaf- and internal-labels.

The LBT’s with unique labels are also defined recursively.
Definition. Base case: The LBT \( \{l, \text{leaf} \} \) has unique labels.

Constructor case: If \( B \) and \( C \) are LBT’s with unique labels, no label of \( B \) is a label \( C \) and vice-versa, and \( l \) is not a label of \( B \) or \( C \), then \( \{l, B, C\} \) has unique labels.

If \( B \) is an LBT, let \( n_B \) be the number of distinct internal-labels appearing in \( B \) and \( f_B \) be the number of distinct leaf labels of \( B \). Prove by structural induction that

\[
f_B = n_B + 1
\]

for all LBT’s \( B \) with unique labels. This equation can obviously fail if labels are not unique, so your proof had better use uniqueness of labels at some point; be sure to indicate where.

Problem 3.

In this problem you will prove a fact that may surprise you—or make you even more convinced that set theory is nonsense: the half-open unit interval is actually the “same size” as the nonnegative quadrant of the real plane!¹ Namely, there is a bijection from \((0, 1)\) to \([0, \infty) \times [0, \infty)\).

(a) Describe a bijection from \((0, 1)\) to \([0, \infty)\).

Hint: \( 1/x \) almost works.

(b) An infinite sequence of the decimal digits \( \{0, 1, \ldots, 9\} \) will be called long if it does not end with all 0’s. An equivalent way to say this is that a long sequence is one that has infinitely many occurrences of nonzero digits. Let \( L \) be the set of all such long sequences. Describe a bijection from \( L \) to the half-open real interval \((0, 1]\).

Hint: Put a decimal point at the beginning of the sequence.

(c) Describe a surjective function from \( L \) to \( L^2 \) that involves alternating digits from two long sequences.

Hint: The surjection need not be total.

(d) Prove the following lemma and use it to conclude that there is a bijection from \( L^2 \) to \((0, 1]^2\).

Lemma 3.1. Let \( A \) and \( B \) be nonempty sets. If there is a bijection from \( A \) to \( B \), then there is also a bijection from \( A \times A \) to \( B \times B \).

(e) Conclude from the previous parts that there is a surjection from \((0, 1] \) to \((0, 1]^2\). Then appeal to the Schröder-Bernstein Theorem to show that there is actually a bijection from \((0, 1] \) to \((0, 1]^2\).

(f) Complete the proof that there is a bijection from \((0, 1] \) to \([0, \infty)^2\).

¹The half-open unit interval, \((0, 1]\), is \( \{r \in \mathbb{R} \mid 0 < r \leq 1\} \). Similarly, \([0, \infty) := \{r \in \mathbb{R} \mid r \geq 0\} \).