

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science
6.042J/18.062J

The Logic of Propositions



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propositional logic.1

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Proving Validity

Instead of truth tables,
can try to **prove** valid
formulas symbolically using
axioms and deduction rules



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Proving Validity

The text describes a
bunch of algebraic rules to
prove that propositional
formulas are equivalent



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Algebra for Equivalence

for example,
the **distributive law**

$$P \text{ AND } (Q \text{ OR } R) \equiv (P \text{ AND } Q) \text{ OR } (P \text{ AND } R)$$



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Algebra for Equivalence

for example,
DeMorgan's law

$$\text{NOT}(P \text{ AND } Q) \equiv \text{NOT}(P) \text{ OR } \text{NOT}(Q)$$



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Algebra for Equivalence

The set of rules for \equiv in the text are **complete**: if two formulas are \equiv , these rules can prove it.



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A Proof System

Another approach is to start with some valid formulas (**axioms**) and deduce more valid formulas using **proof rules**



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A Proof System

Lukasiewicz' proof system is a particularly elegant example of this idea.



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A Proof System

Lukasiewicz' proof system is a particularly elegant example of this idea. It covers formulas whose only logical operators are **IMPLIES** (\rightarrow) and **NOT**.



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Lukasiewicz' Proof System

Axioms:

- 1) $(\neg P \rightarrow P) \rightarrow P$
- 2) $P \rightarrow (\neg P \rightarrow Q)$
- 3) $(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$

The only rule: **modus ponens**



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Lukasiewicz' Proof System

Prove formulas by starting with axioms and repeatedly applying the inference rule.

To illustrate the proof system we'll do an example, which you may safely skip.



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Lukasiewicz' Proof System

Prove formulas by starting with axioms and repeatedly applying the inference rule.

For example, to prove:

$$P \rightarrow P$$



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A Lukasiewicz' Proof

3rd axiom:

$$(P \rightarrow Q) \rightarrow$$

$$((Q \rightarrow R) \rightarrow (P \rightarrow R))$$

replace R by P



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A Lukasiewicz' Proof

3rd axiom:

$$(P \rightarrow Q) \rightarrow$$

$$((Q \rightarrow P) \rightarrow (P \rightarrow P))$$

replace Q by $(\bar{P} \rightarrow P)$



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A Lukasiewicz' Proof

3rd axiom:

Axiom 2)

$$(P \rightarrow (\bar{P} \rightarrow P)) \rightarrow$$

$$(((\bar{P} \rightarrow P) \rightarrow P) \rightarrow (P \rightarrow P))$$



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A Lukasiewicz' Proof

so apply modus ponens:

Axiom 2)

$$(P \rightarrow (\bar{P} \rightarrow P)) \rightarrow$$

$$(((\bar{P} \rightarrow P) \rightarrow P) \rightarrow (P \rightarrow P))$$



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A Lukasiewicz' Proof

so apply **modus ponens**:

$$\overbrace{(((\bar{P} \rightarrow P) \rightarrow P) \rightarrow (P \rightarrow P))}^{\text{Axiom 1)}}$$



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A Lukasiewicz' Proof

so apply **modus ponens**:

$$(P \rightarrow P)$$

QED



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Lukasiewicz is **Sound**

The 3 Axioms are all **valid**
(verify by truth table).

We know modus ponens is **sound**. So **every provable formula is also valid**.



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Lukasiewicz is **Complete**

Conversely, **every valid (NOT, →)-formula is provable**:

system is "complete"

Not hard to verify but would take a full lecture; we omit it.



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validity checking still inefficient

Algebraic & deduction
proofs **in general** are no
better than truth tables.
**No efficient method for
verifying validity is known.**



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