Midterm Exam March 18

Your name:____

Circle your **Session:** **1** **2:30**
Table: **A B C D E F G H I J K**

- This exam is **closed book** except for a 2-sided cribsheet. Total time is 80 minutes.
- Write your solutions in the space provided. If you need more space, **write on the back** of the sheet containing the problem.
- In answering the following questions, you may use without proof any of the results from class or text.

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	15		
2	20		
3	10		
4	20		
5	15		
6	20		
Total	100		

Problem 1 (**Structural Induction**) (**15 points**).

Definition. The set RAF of *rational functions* of one real variable is the set of functions defined recursively as follows:

Base cases:

- The identity function, $\text{id}(r) ::= r$ for $r \in \mathbb{R}$ (the real numbers), is an RAF,
- any constant function on $\mathbb{R}$ is an RAF.

Constructor cases: If f, g are RAF's, then so is $f \circledast g$, where $\circledast$ is one of the operations

1. addition, $+$,
2. multiplication, $\cdot$, and
3. division $/$.

Prove by structural induction that RAF is closed under composition. That is, using the induction hypothesis,

$$P(h) ::= \forall g \in \text{RAF} \,.\, h \circ g \in \text{RAF}, \tag{1}$$

prove that $P(h)$ holds for all $h \in \text{RAF}$. Make sure to indicate explicitly

- each of the base cases, and
- each of the constructor cases. *Hint:* One proof in terms of $\circledast$ covers all three cases.

Problem 2 (State Machines) (20 points).
The Stata Center's delicate balance depends on two buckets of water hidden in a secret room. The big bucket has a volume of 25 gallons, and the little bucket has a volume of 10 gallons. If at any time a bucket contains exactly 13 gallons, the Stata Center will collapse. There is an interactive display where tourists can remotely fill and empty the buckets according to certain rules. We represent the buckets as a state machine.

The state of the machine is a pair (b, l), where b is the volume of water in big bucket, and l is the volume of water in little bucket.

(a) We informally describe some of the legal operations tourists can perform below. Represent each of the following operations as a transition of the state machine. The first is done for you as an example.

1. Fill the big bucket.

$$(b, l) \longrightarrow (25, l).$$

2. Empty the little bucket.

3. Pour the big bucket into the little bucket. You should have two cases defined in terms of the state (b, l): if all the water from the big bucket fits in the little bucket, then pour all the water. If it doesn't, pour until the little jar is full, leaving some water remaining in the big jar.

(b) Use the Invariant Principle to show that, starting with empty buckets, the Stata Center will never collapse. That is, the state $(13, x)$ in unreachable. (In verifying your claim that the invariant is preserved, you may restrict to the representative transitions of part (a).)

Problem 3 (Jections) (10 points).
Prove that if A is an infinite set and B is a countably infinite set that has no elements in common with A, then

$$A \text{ bij } (A \cup B).$$

Reminder: You may assume any of the results from class, MITx, or the text as long as you state them explicitly.

Problem 4 (GCDs) (20 points).
Let

$$
\begin{aligned}
m &= 2^9 5^{24} 7^4 11^7, \\
n &= 2^3 7^{22} 11^{211} 19^7, \\
p &= 2^5 3^4 7^{6042} 19^{30}.
\end{aligned}
$$

(a) What is the $\gcd(m, n, p)$?

(b) What is the *least common multiple*, $\text{lcm}(m, n, p)$?

Let $\nu_k(n)$ be the largest power of k that divides n, where $k > 1$. That is,

$$\nu_k(n) ::= \max\{i \mid k^i \text{ divides } n\}.$$

If A is a nonempty set of nonnegative integers, define

$$\nu_k(A) ::= \{\nu_k(a) \mid a \in A\}.$$

(c) Express $\nu_k(\gcd(A))$ in terms of $\nu_k(A)$.

(d) Let p be a prime number. Express $\nu_p(\text{lcm}(A))$ in terms of $\nu_p(A)$.

(e) Give an example of integers a, b where $\nu_6(\text{lcm}(a, b)) > \max(\nu_6(a), \nu_6(b))$.

Problem 5 (Congruences) (15 points).
Prove that if $a \equiv b \pmod{14}$ and $a \equiv b \pmod{5}$, then $a \equiv b \pmod{70}$.

Problem 6 (Euler's function) (20 points).
Let ϕ be Euler's function.

(a) What is the value of $\phi(2)$?

(b) What are three nonnegative integers $k > 1$ such that $\phi(k) = 2$?

(c) Prove that $\phi(k)$ is even for $k > 2$.

Hint: Consider whether k has an odd prime factor or not.