Induction

The Idea of Induction
Color the integers $\geq 0$
0, 1, 2, 3, 4, 5, ?, ... 
I tell you, 0 is red, & any int next to a red integer is red, then you know that all the ints are red!

Induction Rule
$R(0), R(0) \implies R(1), R(1) \implies R(2), R(2) \implies R(3), \ldots, R(n) \implies R(n+1), \ldots$
$R(0), R(1), R(2), \ldots, R(n), \ldots$
Induction Rule

\[ R(0), \forall n . R(n) \implies R(n+1) \]

\[ \forall m . R(m) \]

Example Induction Proof

Let’s prove:

\[ 1+r+r^2+\ldots+r^n = \frac{r^{(n+1)}-1}{r-1} \]

(for \( r \neq 1 \))

Like Dominos...

Statements in magenta form a template for inductive proofs:

Proof: (by induction on \( n \))

The induction hypothesis, \( P(n) \), is:

\[ 1+r+r^2+\ldots+r^n = \frac{r^{(n+1)}-1}{r-1} \]

(for \( r \neq 1 \))
Example Induction Proof

Base Case \((n = 0)\):\n\[
1 + r + r^2 + \ldots + r^0 = \frac{r^{0+1} - 1}{r - 1}
\]
\[
= \frac{r - 1}{r - 1} = 1
\]
OK!

Inductive Step: Assume \(P(n)\) where \(n \geq 0\) and prove \(P(n+1)\):\n\[
1 + r + r^2 + \ldots + r^{n+1} = \frac{r^{(n+1)+1} - 1}{r - 1}
\]

Now from induction hypothesis \(P(n)\) we have\n\[
1 + r + r^2 + \ldots + r^n = \frac{r^{n+1} - 1}{r - 1}
\]
so add \(r^{n+1}\) to both sides\n\[
(1 + r + r^2 + \ldots + r^n) + r^{n+1} = \frac{r^{n+1} - 1}{r - 1} + r^{n+1}
\]
This proves \(P(n+1)\) completing the proof by induction.
an aside: ellipsis

“…” is an ellipsis. Means you should see a pattern:

\[ 1 + r + r^2 + \ldots + r^n = \sum_{i=0}^{n} r^i \]

Can lead to confusion (n = 0?)

sum (Σ) notation more precise
Gehry specifies L-shaped tiles covering three squares:

For example, for 8 x 8 plaza might tile for Bill this way:

Theorem: For any $2^n \times 2^n$ plaza, we can make Bill and Frank happy.

Proof: (by induction on $n$)

$P(n) ::= \text{can tile } 2^n \times 2^n \text{ with Bill in middle.}$

Base case: ($n=0$)

(no tiles needed)

Induction step: assume can tile $2^n \times 2^n$, prove can tile $2^{n+1} \times 2^{n+1}$.
The fix: prove something \textbf{stronger}—that we can find a tiling with Bill in \textbf{any} square.

Theorem: For any \(2^n \times 2^n\) plaza, we can make Bill and Frank happy.

Proof: (by \textit{induction on} \(n\))

\begin{itemize}
  \item \textbf{Base case:} \((n=0)\) as before
  \item \textbf{Inductive step:}
    \begin{itemize}
      \item \textit{Assume} we can get Bill \textbf{anywhere} in \(2^n \times 2^n\)
      \item \textit{Prove} we can get Bill \textbf{anywhere} in \(2^{n+1} \times 2^{n+1}\)
    \end{itemize}
\end{itemize}

Now group the squares together, and fill the center Bill’s with a tile.

Done!
Recursive Procedure

**Note**: The induction proof implicitly defines a recursive procedure for tiling with Bill anywhere.