Induction

The Idea of Induction

Color the integers ≥ 0

0, 1, 2, 3, 4, 5, ?, …

I tell you, 0 is red, & any int next to a red integer is red,

then you know that

all the ints are red!

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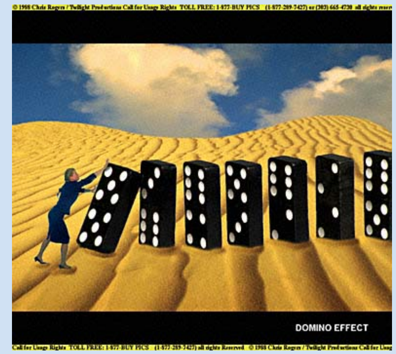
Induction Rule

$$\frac{R(0),\ R(0) \text{ IMPLIES } R(1),\ R(1) \text{ IMPLIES } R(2),\ R(2) \text{ IMPLIES } R(3), \ldots, R(n) \text{ IMPLIES } R(n+1), \ldots}{R(0), R(1), R(2), \ldots, R(n), \ldots}$$

Induction Rule

$$\frac{R(0),\ \forall n.\, R(n) \text{ IMPLIES } R(n+1)}{\forall m.\, R(m)}$$

Like Dominos…



Example Induction Proof

Let's prove:

$$1+r+r^2+\cdots+r^n = \frac{r^{(n+1)}-1}{r-1}$$

(for $r \neq 1$)

Example Induction Proof

Statements in magenta form a **template for inductive proofs:**

Proof: (by induction on n)

The induction hypothesis, P(n), is:

$$1+r+r^2+\cdots+r^n = \frac{r^{(n+1)}-1}{r-1}$$

(for $r \neq 1$)

Example Induction Proof

Base Case ($n = 0$):

$$\underbrace{1+r+r^2+\cdots+r^0}_{1} \overset{?}{=} \frac{r^{0+1}-1}{r-1}$$

$$= \frac{r-1}{r-1} = 1$$

OK!

Example Induction Proof

Inductive Step: Assume $P(n)$ where $n \geq 0$ and prove $P(n+1)$:

$$1+r+r^2+\cdots+r^{n+1} = \frac{r^{(n+1)+1}-1}{r-1}$$

Example Induction Proof

Now from induction hypothesis $P(n)$ we have

$$1+r+r^2+\cdots+r^n = \frac{r^{n+1}-1}{r-1}$$

so add r^{n+1} to both sides

Example Induction Proof

adding r^{n+1} to both sides,

$$\left(1+r+r^2+\cdots+r^n\right)+r^{n+1} = \left(\frac{r^{n+1}-1}{r-1}\right)+r^{n+1}$$

$$= \frac{r^{n+1}-1+r^{n+1}(r-1)}{r-1}$$

$$= \frac{r^{(n+1)+1}-1}{r-1}$$

This proves $P(n+1)$ completing the proof by induction.

an aside: ellipsis

"…" is an ellipsis. Means you should see a pattern:

$$1 + r + r^2 + \cdots + r^n = \sum_{i=0}^{n} r^i$$

Can lead to confusion (n = 0?)

sum (Σ) notation more precise

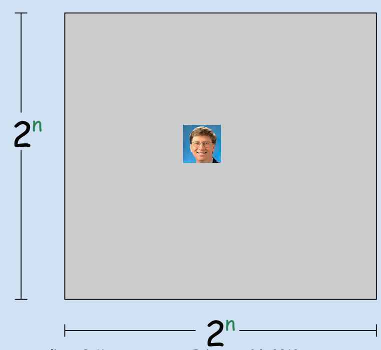
The MIT Stata Center

Design Mockup: Stata Lobby

Mockup: Plaza Outside Stata

Goal: Tile the plaza, except for 1×1 square in the middle for Bill.

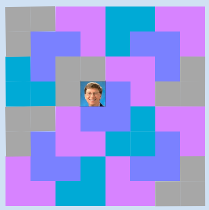
(Picture source: http://www.microsoft.com/presspass/exec/billg/default.asp)



Plaza Outside Stata

Gehry specifies L-shaped tiles covering three squares:

For example, for 8 x 8 plaza might tile for Bill this way:



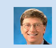
Plaza Outside Stata

Theorem: For any $2^n \times 2^n$ plaza, we can make Bill and Frank happy.

Proof: (by induction on n)

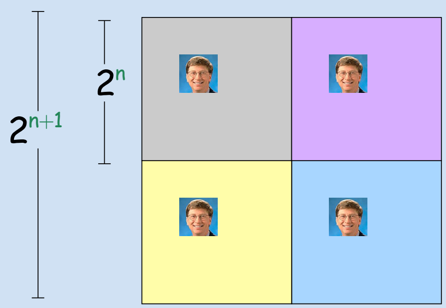
$P(n)$ ::= can tile $2^n \times 2^n$ with Bill in middle.

Base case: ($n=0$)

 (no tiles needed)

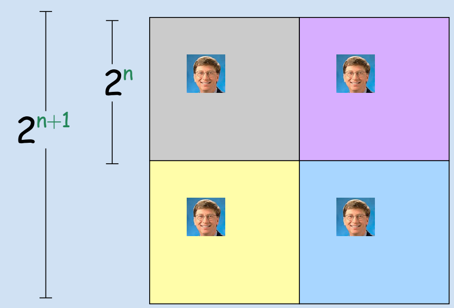
Plaza Outside Stata

Induction step: assume can tile $2^n \times 2^n$, prove can tile $2^{n+1} \times 2^{n+1}$



Plaza Outside Stata

Now what?...



Plaza Outside Stata

The fix:

prove something stronger —that we can find a tiling with Bill in any square.

Plaza Theorem

Theorem: For any $2^n \times 2^n$ plaza, we can make Bill and Frank happy.

Proof: (by induction on n)
revised induction hypothesis $P(n)$ ::=
can tile with Bill anywhere

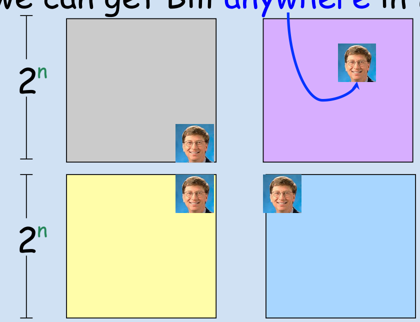
Base case: $(n=0)$ as before

Plaza Proof

Inductive step:

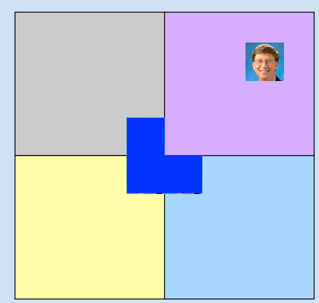
Assume we can get Bill anywhere in $2^n \times 2^n$

Prove we can get Bill anywhere in $2^{n+1} \times 2^{n+1}$



Plaza Proof

Now group the squares together, and fill the center Bill's with a tile.



Done!

Recursive Procedure

Note: The induction proof implicitly defines a

recursive procedure

for tiling with Bill anywhere.