

6	9	13	7
12		10	5
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# Induction



Albert R Meyer

February 24, 2012

lec 3F.1

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## The Idea of Induction

Color the integers  $\geq 0$

0, 1, 2, 3, 4, 5, ?, ...

I tell you, 0 is red, & any int next to a red integer is red, then you know that all the ints are red!



Albert R Meyer

February 24, 2012

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6	9	13	7
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## The Idea of Induction

Color the integers  $\geq 0$

0, 1, 2, 3, 4, 5, ...

I tell you, 0 is red, & any int next to a red integer is red, then you know that all the ints are red!



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## Induction Rule

$R(0), R(0) \text{ IMPLIES } R(1), R(1) \text{ IMPLIES } R(2),$   
 $R(2) \text{ IMPLIES } R(3), \dots, R(n) \text{ IMPLIES } R(n+1), \dots$

$R(0), R(1), R(2), \dots, R(n), \dots$



Albert R Meyer

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## Induction Rule

$$R(0), \forall n. R(n) \text{ IMPLIES } R(n+1)$$


---


$$\forall m. R(m)$$

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## Like Dominos...

DOMINO EFFECT

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## Example Induction Proof

Let's prove:

$$1+r+r^2+\dots+r^n = \frac{r^{(n+1)}-1}{r-1}$$

(for  $r \neq 1$ )

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## Example Induction Proof

Statements in **magenta** form a  
**template for inductive proofs:**  
**Proof:** (by induction on  $n$ )  
 The induction hypothesis,  $P(n)$ , is:

$$1+r+r^2+\dots+r^n = \frac{r^{(n+1)}-1}{r-1}$$

(for  $r \neq 1$ )

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### Example Induction Proof

Base Case ( $n = 0$ ):

$$\underbrace{1+r+r^2+\dots+r^0}_{1} = \overset{?}{\frac{r^{0+1}-1}{r-1}} = \frac{r-1}{r-1} = 1$$

OK!


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### Example Induction Proof

Inductive Step: Assume  $P(n)$  where  $n \geq 0$  and prove  $P(n+1)$ :

$$1+r+r^2+\dots+r^{n+1} = \frac{r^{(n+1)+1}-1}{r-1}$$

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### Example Induction Proof

Now from induction hypothesis  $P(n)$  we have

$$1+r+r^2+\dots+r^n = \frac{r^{n+1}-1}{r-1}$$

so add  $r^{n+1}$  to both sides

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### Example Induction Proof

adding  $r^{n+1}$  to both sides,

$$(1+r+r^2+\dots+r^n) + r^{n+1} = \left(\frac{r^{n+1}-1}{r-1}\right) + r^{n+1}$$

This proves  $P(n+1)$  completing the proof by induction.

$$= \frac{r^{n+1}-1+r^{n+1}(r-1)}{r-1} = \frac{r^{(n+1)+1}-1}{r-1}$$

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**an aside: ellipsis**

"..." is an **ellipsis**. Means you should **see a pattern**:

$$1 + r + r^2 + \dots + r^n = \sum_{i=0}^n r^i$$

Can lead to confusion (n = 0?)

sum ( $\Sigma$ ) notation more precise


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## The MIT Stata Center



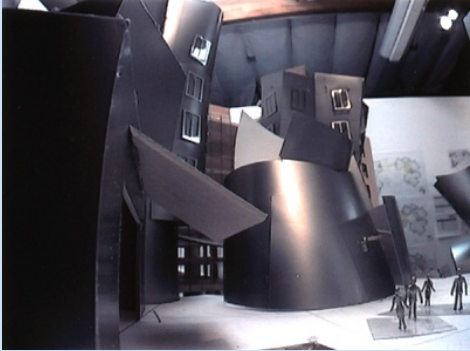
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## Design Mockup: Stata Lobby



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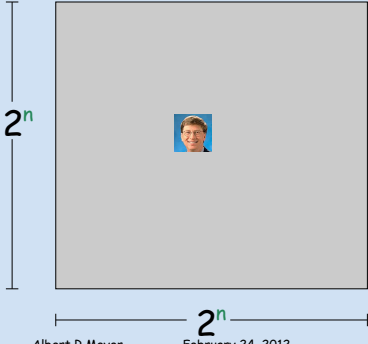
lec 3F.15

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## Mockup: Plaza Outside Stata

Goal: Tile the plaza, except for 1x1 square in the middle for Bill.

(Picture source: <http://www.microsoft.com/presspass/exec/billg/default.asp>)



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## Plaza Outside Stata

Gehry specifies L-shaped tiles covering three squares:

For example, for 8 x 8 plaza might tile for Bill this way:

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## Plaza Outside Stata

Theorem: For any  $2^n \times 2^n$  plaza, we can make Bill and Frank happy.

Proof: (by induction on  $n$ )  
 $P(n) ::=$  can tile  $2^n \times 2^n$  with Bill in middle.

Base case: ( $n=0$ )

(no tiles needed)

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## Plaza Outside Stata

Induction step: assume can tile  $2^n \times 2^n$ , prove can tile  $2^{n+1} \times 2^{n+1}$

$2^{n+1}$

$2^n$

$2^n$

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## Plaza Outside Stata

Now what?...

$2^{n+1}$

$2^n$

$2^n$

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## Plaza Outside Stata

The fix:  
prove something stronger  
—that we can find a tiling  
with Bill in any square.

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## Plaza Theorem

Theorem: For any  $2^n \times 2^n$  plaza, we  
can make Bill and Frank happy.

Proof: (by induction on  $n$ )  
revised induction hypothesis  $P(n) ::=$   
can tile with Bill anywhere

Base case: ( $n=0$ ) as before

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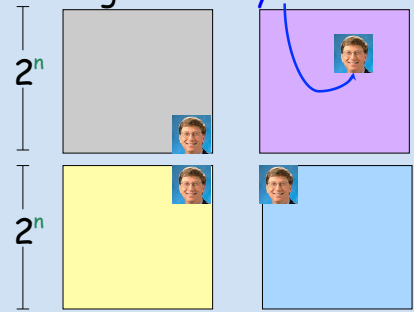
6	9	13	7
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## Plaza Proof

Inductive step:

Assume we can get Bill anywhere in  $2^n \times 2^n$

Prove we can get Bill anywhere in  $2^{n+1} \times 2^{n+1}$

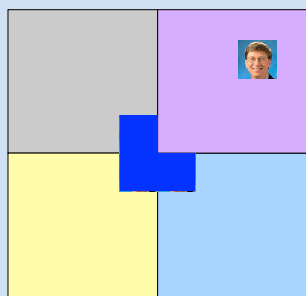


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## Plaza Proof

Now group the squares together,  
and fill the center Bill's with a tile.



Done!

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## Recursive Procedure

**Note:** The induction proof implicitly defines a recursive procedure for tiling with Bill anywhere.

