

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science

MIT 6.042J/18.062J

Finite Cardinality



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finite-card.1

6	9	13	7
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Mapping Rule (bij)

A bijection from
A to B implies

$$|A| = |B|$$

for finite A, B



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6	9	13	7
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size of the power set

subsets of a finite set A?

$$|\text{pow}(A)| ?$$

for $A = \{a, b, c\}$, $\text{pow}(A) =$

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \\ \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$



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6	9	13	7
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$\text{pow}(A)$ bijection to bit-strings

A: $\{a_0, a_1, a_2, a_3, a_4, \dots, a_{n-1}\}$

subset: $\{a_0, a_2, a_3, \dots, a_{n-1}\}$

string: 1 0 1 1 0 ... 1

this defines a bijection, so

$$\# \text{ n-bit strings} = |\text{pow}(A)|$$



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6	9	13	7
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pow(A) bijection to bit-strings

every computer scientist knows #n-bit strings, so Corollary:

$$|\text{pow}(A)| = 2^{|A|}$$



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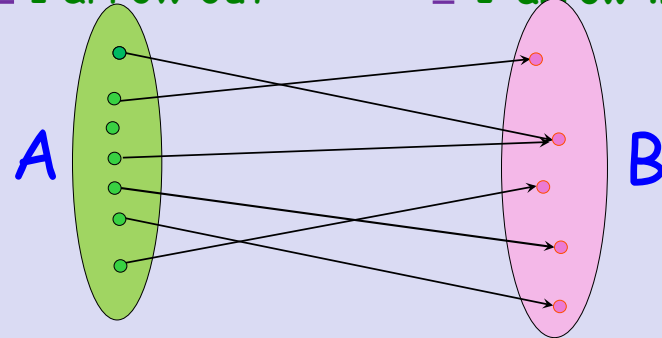
finite-card.5

6	9	13	7
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function & surjective

≤ 1 arrow out

≥ 1 arrow in



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finite-card.6

6	9	13	7
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Mapping Rule (surj)

function: $A \rightarrow B$



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finite-card.7

6	9	13	7
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Mapping Rule (surj)

$[\leq 1 \text{ out}]$: $A \rightarrow B$

IMPLIES $|A| \geq \#\text{arrows}$.

surjection: $A \rightarrow B$



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finite-card.8

6	9	13	7
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Mapping Rule (surj)

$[\leq 1 \text{ out}] : A \rightarrow B$
 IMPLIES $|A| \geq \#\text{arrows}$.
 $[\geq 1 \text{ in}] : A \rightarrow B$
 IMPLIES $\#\text{arrows} \geq |B|$.



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finite-card.9

6	9	13	7
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Mapping Rule (surj)

Surjective function
 from A to B implies
 $|A| \geq |B|$
 for finite A, B



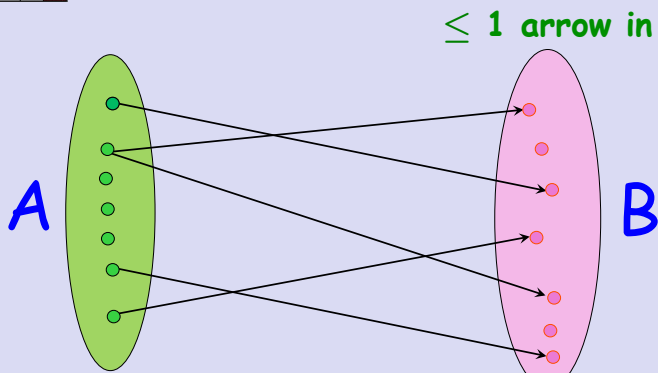
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finite-card.10

6	9	13	7
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injection archery



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finite-card.11

6	9	13	7
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Mapping Rule (inj)

total $[\geq 1 \text{ out}]$ IMPLIES
 $|A| \leq \#\text{arrows}$
 injection $[\leq 1 \text{ in}]$ IMPLIES
 $\#\text{arrows} \leq |B|$



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finite-card.12

6	9	13	7
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Mapping Rule (inj)

Total injective relation
from A to B implies
 $|A| \leq |B|$
for finite A, B



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finite-card.13

6	9	13	7
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"jection" relations

$A \text{ bij } B ::= \exists \text{bijection: } A \rightarrow B$
 $A \text{ surj } B ::= \exists \text{surj func: } A \rightarrow B$
 $A \text{ inj } B ::= \exists \text{total inj}$
 relation: $A \rightarrow B$



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finite-card.14

6	9	13	7
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Mapping Lemma

$A \text{ bij } B$ IFF $|A| = |B|$
 $A \text{ surj } B$ IFF $|A| \geq |B|$
 $A \text{ inj } B$ IFF $|A| \leq |B|$
 for finite A, B



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finite-card.15

6	9	13	7
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Familiar "size" properties

$|A| = |B| = |C|$ IMPLIES $|A| = |C|$
 $|A| \geq |B| \geq |C|$ IMPLIES $|A| \geq |C|$
 $|A| \geq |B| \geq |A|$ IMPLIES $|A| = |B|$
 for finite A, B, C



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6	9	13	7
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Familiar "size" properties

$A \text{ bij } B \text{ bij } C \text{ IMPLIES } A \text{ bij } C$
 $A \text{ surj } B \text{ surj } C \text{ IMPLIES } A \text{ surj } C$
 $A \text{ surj } B \text{ surj } A \text{ IMPLIES } A \text{ bij } B$

for **finite** A, B, C
by the Mapping Lemma



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6	9	13	7
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Familiar "size" properties

$A \text{ bij } B \text{ bij } C \text{ IMPLIES } A \text{ bij } C$
 $A \text{ surj } B \text{ surj } C \text{ IMPLIES } A \text{ surj } C$
 $A \text{ surj } B \text{ surj } A \text{ IMPLIES } A \text{ bij } B$

for **infinite** A, B, C , also
 1st two implications: easy
 3rd is **tricky**



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