Conflict Final 2

Your name:____________________________________

Circle your	**Session:**	**1**	**2:30**									
	Table:	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**

- This exam is **closed book** except for two 2-sided cribsheets. Total time is 180 minutes.
- Write your solutions in the space provided. If you need more space, **write on the back** of the sheet containing the problem.
- In answering the following questions, you may use without proof any of the results from class or text.
- Incorrect short answers are eligible for part credit when there is an explanation.

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	6		
2	10		
3	8		
4	8		
5	8		
6	6		
7	10		
8	10		
9	6		
10	10		
11	12		
12	6		
Total	100		

Problem 1 (Probable Satisfiability) (6 points).
Truth values for propositional variables P, Q, R are chosen independently, with

$$\Pr[P = \mathbf{T}] = 1/2, \ \Pr[Q = \mathbf{T}] = 1/3, \ \Pr[R = \mathbf{T}] = 1/5.$$

What is the probability that the formula

$$P \text{ IMPLIES } (Q \text{ IMPLIES } R)$$

is true? __________

Problem 2 (Induction, Trees) (10 points).

A simple graph, G, is said to have *width* 1 iff there is a way to list all its vertices so that each vertex is adjacent to at most one vertex that appears earlier in the list.

Prove that every finite graph with width one is a forest.

Hint: By induction, removing the last vertex.

Problem 3 (Number Theory) (8 points).
Indicate whether the following statements are **true** or **false**. For each of the false statements, **give counterexamples**. All variables range over the integers, $\mathbb{Z}$.

(a) For all a and b, there are x and y such that: $ax + by = 1$.

(b) $\gcd(mb + r, b) = \gcd(r, b)$ for all m, r and b.

(c) $k^{p-1} \equiv 1 \pmod{p}$ for every prime p and every k.

(d) For primes $p \neq q$, $\phi(pq) = (p-1)(q-1)$, where ϕ is Euler's totient function.

(e) If a and b are relatively prime to d, then

$$[ac \equiv bc \bmod d] \quad \text{IMPLIES} \quad [a \equiv b \bmod d].$$

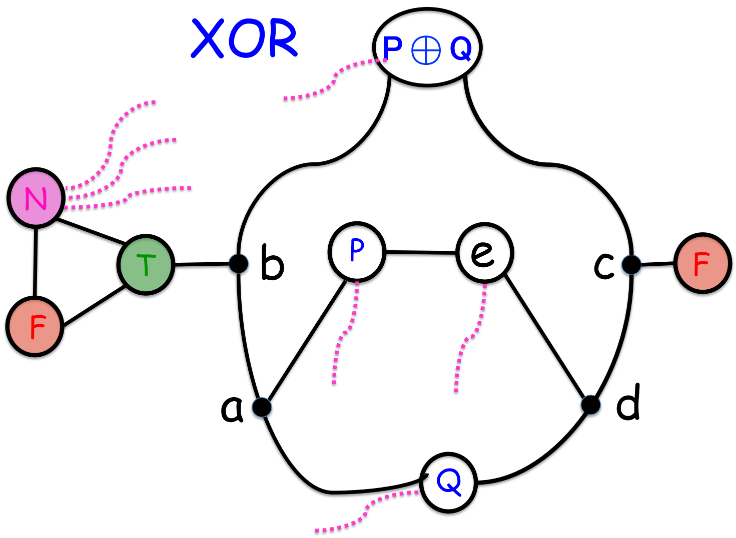

Figure 1 A 3-color XOR-gate

Problem 4 (3-Coloring) (8 points).
In the graph shown in Figure 1, the vertices connected in the triangle on the left are called *color-vertices*; since they form a triangle, they are forced to have different colors in any coloring of the graph. The colors assigned to the color-vertices will be called **T**, **F** and **N**. The dotted lines indicate edges to the color-vertex **N**.

Carefully explain why, for any assignment of *different* truth-colors to P and Q, there is a *unique* 3-coloring of the graph.

Problem 5 (**Stable Marriage**) (**8 points**).

We are interested in invariants of the Mating Ritual for finding stable marriages. Let Angelina and Jen be two of the girls, and Keith and Tom be two of the boys.

Which of the following predicates are preserved invariants of the Mating Ritual no matter what the preferences are among the boys and girls?

(a) Tom is not serenading Jen.

(b) Tom's list of girls to serenade is empty.

(c) All the boys have the same number of girls left uncrossed in their lists.

(d) Jen is crossed off Keith's list and Keith prefers Jen to anyone he is serenading.

Problem 6 (**Big Oh**) (**6 points**).
Define two functions f, g that are incomparable under big Oh:

$$f \neq O(g) \text{ AND } g \neq O(f).$$

Problem 7 (Counting) (10 points).
In a standard 52-card deck (13 ranks and 4 suits), a hand is a 5-card subset of the set of 52 cards. Express the answer to each part as a formula using factorial, binomial, or multinomial notation.

(a) Let H_{NP} be the set of all hands that include no pairs; that is, no two cards in the hand have the same rank.

What is $|H_{NP}|$? ____________

(b) Let H_S be the set of all hands that are straights, i.e. the ranks of the five cards are consecutive. The order of the ranks is $(A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, k, A)$; note that A appears twice.

What is $|H_S|$? ____________

(c) Let H_F be the set of all hands that are flushes; that is, the suits of the five cards are identical.

What is $|H_F|$? ____________

(d) Let H_{SF} be the set of all straight flush hands; that is, the hand is both a straight and a flush.

What is $|H_{SF}|$? ____________

(e) Let H_{HC} be the set of all high-card hands; that is, hands that do not include pairs, are not straights, and are not flushes.

What is $|H_{HC}|$? ____________

Problem 8 (**Conditional Probability**) (**10 points**).

Here's a variation of Monty Hall's game: the contestant still picks one of three doors, with a prize randomly placed behind one door and goats behind the other two. But now, instead of always opening a door to reveal a goat, Monty instructs Carol to *randomly* open one of the two doors that the contestant hasn't picked. This means she may reveal a goat, or she may reveal the prize. If she reveals the prize, then the entire game is *restarted*, that is, the prize is again randomly placed behind some door, the contestant again picks a door, and so on until Carol finally picks a door with a goat behind it. Then the contestant can choose to *stick* with his original choice of door or *switch* to the other unopened door. He wins if the prize is behind the door he finally chooses.

To analyze this setup, we define two events:

GP**:** The event that the contestant **g**uesses the door with the **p**rize behind it on his first guess.

OP**:** The event that the game is restarted at least once. Another way to describe this is as the event that the door Carol first **o**pens has a **p**rize behind it.

Give the values of the following probabilities:

(a) $\Pr\left[OP \mid \overline{GP}\right]$ __________

(b) $\Pr[OP]$ __________

(c) the probability that the game will continue forever __________

(d) When Carol finally picks the goat, the contestant has the choice of sticking or switching. Let's say that the contestant adopts the strategy of sticking. Let W be the event that the contestant wins with this strategy, and let $w ::= \Pr[W]$. Express the following conditional probabilities as simple closed forms in terms of w.

i) $\Pr\left[W \mid GP\right]$ __________

ii) $\Pr\left[W \mid \overline{GP} \cap OP\right]$ __________

iii) $\Pr\left[W \mid \overline{GP} \cap \overline{OP}\right]$ __________

(e) What is the value of $\Pr[W]$? __________

(f) Let R be the number of times the game is restarted before Carol picks a goat.

What is $\mathrm{Ex}[R]$? __________

(You may express the answer as a simple closed form in terms of $p ::= \Pr[OP]$.)

Problem 9 (**Expectation**) (**6 points**).

A simple graph with n vertices is constructed by randomly placing an edge between every two vertices with probability p. These random edge placements are performed independently.

(a) What is the probability that a given vertex of the graph has degree two?

(b) What is the expected number of nodes with degree two? (You may express your answer in terms of t, where t is the answer to part (a).)

Problem 10 (**Variance, Sums**) (**10 points**).
You have a coin with probability p of flipping heads. For your first try, you flip it once. For your second try, you independently flip it twice. You continue until the nth try, where you independently flip it n times. You *win* a try if you flip all heads. Let W be the number of winning tries. Write a closed-form expression for $\text{Var}[W]$.

Problem 11 (**Markov & Chebyshev**) (**12 points**).
Albert has a gambling problem. He plays 240 hands of draw poker, 120 hands of black jack, and 40 hands of stud poker per day. He wins a hand of draw poker with probability 1/6, a hand of black jack with probability 1/2, and a hand of stud poker with probability 1/5. Let W be the expected number of hands that Albert wins in a day.

(a) What is $\text{Ex}[W]$?

(b) What would the Markov bound be on the probability that Albert will win at least 216 hands on a given day?

(c) Assume the outcomes of the card games are pairwise independent. What is $\text{Var}[W]$? You may answer with a numerical expression that is not completely evaluated.

(d) What would the Chebyshev bound be on the probability that Albert will win at least 216 hands on a given day? You may answer with a numerical expression that includes the constant $v = \text{Var}[W]$.

Problem 12 (**Random Walks**) (**6 points**).
Give simple examples of random walk graphs with the following properties.

(a) A graph with an uncountable number of stationary distributions.

(b) A graph with unique stationary distribution that is not strongly connected.

(c) A strongly connected graph with an initial distribution that does not converge to the stationary distribution.