In-Class Problems Week 9, Fri.

Problem 1.
We begin with two large glasses. The first glass contains a pint of water, and the second contains a pint of wine. We pour 1/3 of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour 1/3 of a pint of the mix back into the first glass and repeat this pouring back-and-forth process a total of n times.

(a) Describe a closed-form formula for the amount of wine in the first glass after n back-and-forth pourings.

(b) What is the limit of the amount of wine in each glass as n approaches infinity?

Problem 2.
An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine d days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to make multiple trips carrying up to a gallon each time to create water caches out in the desert.

For example, if the shrine were $2/3$ of a day's walk into the desert, then she could recover the Holy Grail after two days using the following strategy. She leaves the oasis with 1 gallon of water, travels $1/3$ day into the desert, caches $1/3$ gallon, and then walks back to the oasis—arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks $1/3$ day into the desert, tops off her water supply by taking the $1/3$ gallon in her cache, walks the remaining $1/3$ day to the shrine, grabs the Holy Grail, and then walks for $2/3$ of a day back to the oasis—again arriving with no water to spare.

But what if the shrine were located farther away?

(a) What is the most distant point that the explorer can reach and then return to the oasis, with no water precached in the desert, if she takes a total of only 1 gallon from the oasis?

(b) What is the most distant point the explorer can reach and still return to the oasis if she takes a total of only 2 gallons from the oasis? No proof is required; just do the best you can.

(c) The explorer will travel using a recursive strategy to go far into the desert and back, drawing a total of n gallons of water from the oasis. Her strategy is to build up a cache of $n-1$ gallons, plus enough to get home, a certain fraction of a day's distance into the desert. On the last delivery to the cache, instead of returning home, she proceeds recursively with her $n-1$ gallon strategy to go farther into the desert and return to the cache. At this point, the cache has just enough water left to get her home.

Prove that with n gallons of water, this strategy will get her $H_n/2$ days into the desert and back, where H_n is the nth Harmonic number:

$$H_n ::= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Conclude that she can reach the shrine, however far it is from the oasis.

(d) Suppose that the shrine is $d = 10$ days walk into the desert. Use the asymptotic approximation $H_n \sim \ln n$ to show that it will take more than a million years for the explorer to recover the Holy Grail.

Problem 3.
Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a weakly decreasing function. Define

$$S ::= \sum_{i=1}^{n} f(i)$$

and

$$I ::= \int_{1}^{n} f(x)\, dx.$$

Prove that

$$I + f(n) \le S \le I + f(1).$$

(Proof by very clear picture is OK.)

Problem 4.
Sammy the Shark is a financial service provider who offers loans on the following terms.

- Sammy loans a client m dollars in the morning. This puts the client m dollars in debt to Sammy.
- Each evening, Sammy first charges a service fee which increases the client's debt by f dollars, and then Sammy charges interest, which multiplies the debt by a factor of p. For example, Sammy might charge a "modest" ten cent service fee and 1% interest rate per day, and then f would be 0.1 and p would be 1.01.

(a) What is the client's debt at the end of the first day?

(b) What is the client's debt at the end of the second day?

(c) Write a formula for the client's debt after d days and find an equivalent closed form.

(d) If you borrowed \$10 from Sammy for a year, how much would you owe him?

Supplemental problem

Problem 5.
You've seen this neat trick for evaluating a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \ldots + z^n \\ zS &= z + z^2 + \ldots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \text{ (where } z \neq 1) \end{aligned}$$

Use the same approach to find a closed-form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \ldots + nz^n$$