In-Class Problems Week 8, Wed.

Problem 1.
A researcher analyzing data on heterosexual sexual behavior in a group of m males and f females found that within the group, the male average number of female partners was 10% larger that the female average number of male partners.

(a) Comment on the following claim. "Since we're assuming that each encounter involves one man and one woman, the average numbers should be the same, so the males must be exaggerating."

(b) For what constant c is $m = c \cdot f$?

(c) The data shows that approximately 20% of the females were virgins, while only 5% of the males were. The researcher wonders how excluding virgins from the population would change the averages. If he knew graph theory, the researcher would realize that the nonvirgin male average number of partners will be $x(f/m)$ times the nonvirgin female average number of partners. What is x?

(d) For purposes of further research, it would be helpful to pair each female in the group with a unique male in the group. Explain why this is not possible.

Problem 2. **(a)** Prove that in every simple graph, there are an even number of vertices of odd degree.

(b) Conclude that at a party where some people shake hands, the number of people who shake hands an odd number of times is an even number.

(c) Call a sequence of people at the party a *handshake sequence* if each person in the sequence has shaken hands with the next person, if any, in the sequence.

Suppose George was at the party and has shaken hands with an odd number of people. Explain why, starting with George, there must be a handshake sequence ending with a different person who has shaken an odd number of hands.

Problem 3.
List all the isomorphisms between the two graphs given in Figure 1. Explain why there are no others.

Problem 4.
Which of the items below are simple-graph properties preserved under isomorphism?

(a) The vertices can be numbered 1 through 7.

(b) There is a cycle that includes all the vertices.

(c) There are two degree 8 vertices.

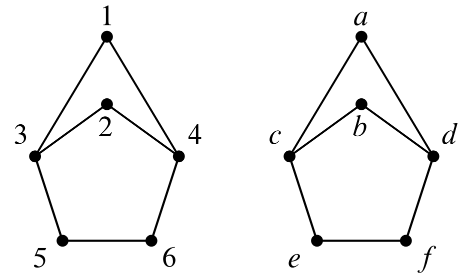

Figure 1 Graphs with several isomorphisms

(d) Two edges are of equal length.

(e) No matter which edge is removed, there is a path between any two vertices.

(f) There are two cycles that do not share any vertices.

(g) One vertex is a subset of another one.

(h) The graph can be pictured in a way that all the edges have the same length.

(i) The OR of two properties that are preserved under isomorphism.

(j) The negation of a property that is preserved under isomorphism.

Supplemental problem

Problem 5.
Let G be a digraph. The neighbors of a vertex v are the endpoints of the edges out of v. Since a digraph is formally the same as a binary relation on $V(G)$, the set of neighbors of v is simply the image, $G(v)$, of v under the relation G.

(a) Suppose f is an isomorphism from G to another digraph H. Prove that

$$f(G(v)) = H(f(v)).$$

Your proof should follow by simple reasoning using the definitions of isomorphism and image of a vertex under the edge relation—no pictures or handwaving.

Hint: Prove by a chain of iff's that

$$h \in H(f(v)) \text{ IFF } h \in f(G(v))$$

for every $h \in V(H)$.

(b) Conclude that if G and H are isomorphic graphs, then they have the same number of vertices of out-degree k, for all $k \in \mathbb{N}$.