In-Class Problems Week 8, Mon.

Problem 1.
For each of the binary relations below, state whether it is a strict partial order, a weak partial order, an equivalence relation, or none of these. If it is a partial order, state whether it is a linear order. If it is none, indicate which of the axioms for partial-order and equivalence relations it violates.

(a) The superset relation $\supseteq$ on the power set pow $\{1, 2, 3, 4, 5\}$.

(b) The relation between any two nonnegative integers a and b such that $a \equiv b \pmod 8$.

(c) The relation between propositional formulas G and H such that [G IMPLIES H] is valid.

(d) The relation between propositional formulas G and H such that [G IFF H] is valid.

(e) The relation 'beats' on Rock, Paper, and Scissors (for those who don't know the game Rock, Paper, Scissors, Rock beats Scissors, Scissors beats Paper, and Paper beats Rock).

(f) The empty relation on the set of real numbers.

(g) The identity relation on the set of integers.

(h) The divisibility relation on the integers, $\mathbb{Z}$.

Problem 2.
The proper subset relation, $\subset$, defines a strict partial order on the subsets of $[1..6]$, that is, on pow($[1..6]$).

(a) What is the size of a maximal chain in this partial order? Describe one.

(b) Describe the largest antichain you can find in this partial order.

(c) What are the maximal and minimal elements? Are they maximum and minimum?

(d) Answer the previous part for the $\subset$ partial order on the set pow $[1..6] - \emptyset$.

Problem 3.
Let S be a sequence of n different numbers. A *subsequence* of S is a sequence that can be obtained by deleting elements of S.

For example, if

$$S = (6, 4, 7, 9, 1, 2, 5, 3, 8)$$

Then 647 and 7253 are both subsequences of S (for readability, we have dropped the parentheses and commas in sequences, so 647 abbreviates $(6, 4, 7)$, for example).

An *increasing subsequence* of S is a subsequence of whose successive elements get larger. For example, 1238 is an increasing subsequence of S. Decreasing subsequences are defined similarly; 641 is a decreasing subsequence of S.

(a) List all the maximum-length increasing subsequences of S, and all the maximum-length decreasing subsequences.

Now let A be the *set* of numbers in S. (So A is the integers $[1..9]$ for the example above.) There are two straightforward linear orders for A. The first is numerical order where A is ordered by the $<$ relation. The second is to order the elements by which comes first in S; call this order $<_S$. So for the example above, we would have

$$6 <_S 4 <_S 7 <_S 9 <_S 1 <_S 2 <_S 5 <_S 3 <_S 8$$

Let $\prec$ be the product relation of the linear orders $<_s$ and $<$. That is, $\prec$ is defined by the rule

$$a \prec a' \quad ::= \quad a < a' \text{ AND } a <_S a'.$$

So $\prec$ is a partial order on A (Section 9.9).

(b) Draw a diagram of the partial order, $\prec$, on A. What are the maximal and minimal elements?

(c) Explain the connection between increasing and decreasing subsequences of S, and chains and anti-chains under $\prec$.

(d) Prove that every sequence, S, of length n has an increasing subsequence of length greater than $\sqrt{n}$ or a decreasing subsequence of length at least $\sqrt{n}$.

Problem 4.

For any total function $f : A \to B$ define a relation $\equiv_f$ by the rule:

$$a \equiv_f a' \quad \text{iff} \quad f(a) = f(a'). \tag{1}$$

(a) Observe (and sketch a proof) that $\equiv_f$ is an equivalence relation on A.

(b) Prove that every equivalence relation, R, on a set, A, is equal to $\equiv_f$ for the function $f : A \to \text{pow}(A)$ defined as

$$f(a) ::= \{a' \in A \mid a \; R \; a'\}.$$

That is, $f(a) = R(a)$.