In-Class Problems Week 7, Fri.

Problem 1.
The table below lists some prerequisite information for some subjects in the MIT Computer Science program (in 2006). This defines an indirect prerequisite relation that is a DAG with these subjects as vertices.

$18.01 \to 6.042$	$18.01 \to 18.02$
$18.01 \to 18.03$	$6.046 \to 6.840$
$8.01 \to 8.02$	$6.001 \to 6.034$
$6.042 \to 6.046$	$18.03, 8.02 \to 6.002$
$6.001, 6.002 \to 6.003$	$6.001, 6.002 \to 6.004$
$6.004 \to 6.033$	$6.033 \to 6.857$

(a) Explain why exactly six terms are required to finish all these subjects, if you can take as many subjects as you want per term. Using a *greedy* subject selection strategy, you should take as many subjects as possible each term. Exhibit your complete class schedule each term using a greedy strategy.

(b) In the second term of the greedy schedule, you took five subjects including 18.03. Identify a set of five subjects not including 18.03 such that it would be possible to take them in any one term (using some nongreedy schedule). Can you figure out how many such sets there are?

(c) Exhibit a schedule for taking all the courses—but only one per term.

(d) Suppose that you want to take all of the subjects, but can handle only two per term. Exactly how many terms are required to graduate? Explain why.

(e) What if you could take three subjects per term?

Problem 2.
A pair of Math for Computer Science Teaching Assistants, Lisa and Annie, have decided to devote some of their spare time this term to establishing dominion over the entire galaxy. Recognizing this as an ambitious project, they worked out the following table of tasks on the back of Annie's copy of the lecture notes.

1. **Devise a logo** and cool imperial theme music - 8 days.
2. **Build a fleet** of Hyperwarp Stardestroyers out of eating paraphernalia swiped from Lobdell - 18 days.
3. **Seize control** of the United Nations - 9 days, after task #1.
4. **Get shots** for Lisa's cat, Tailspin - 11 days, after task #1.
5. **Open a Starbucks chain** for the army to get their caffeine - 10 days, after task #3.

6. **Train an army** of elite interstellar warriors by dragging people to see *The Phantom Menace* dozens of times - 4 days, after tasks #3, #4, and #5.

7. **Launch the fleet** of Stardestroyers, crush all sentient alien species, and establish a Galactic Empire - 6 days, after tasks #2 and #6.

8. **Defeat Microsoft** - 8 days, after tasks #2 and #6.

We picture this information in Figure 1 below by drawing a point for each task, and labelling it with the name and weight of the task. An edge between two points indicates that the task for the higher point must be completed before beginning the task for the lower one.

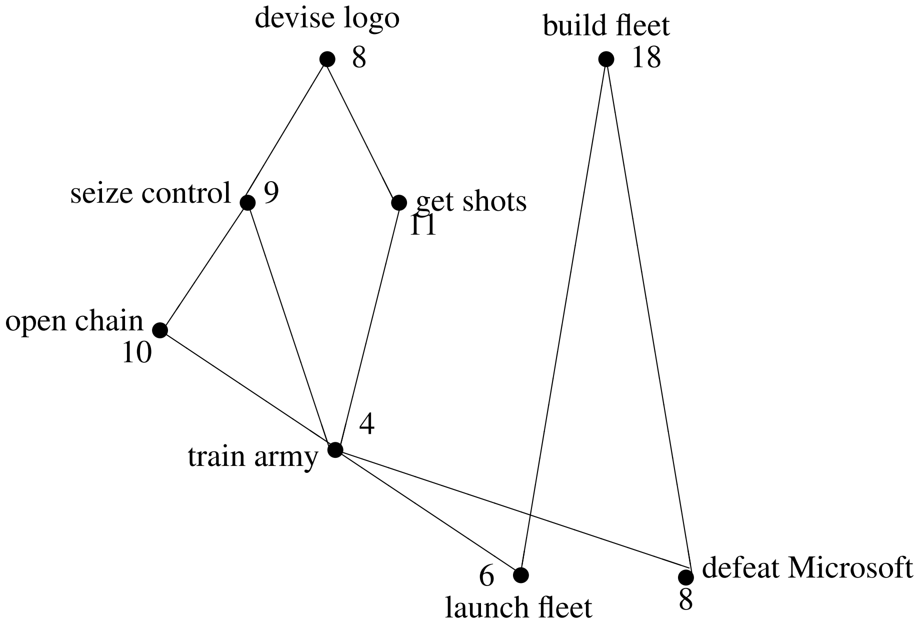

Figure 1 Graph representing the task precedence constraints.

(a) Give some valid order in which the tasks might be completed.

Lisa and Annie want to complete all these tasks in the shortest possible time. However, they have agreed on some constraining work rules.

- Only one person can be assigned to a particular task; they cannot work together on a single task.
- Once a person is assigned to a task, that person must work exclusively on the assignment until it is completed. So, for example, Lisa cannot work on building a fleet for a few days, run to get shots for Tailspin, and then return to building the fleet.

(b) Lisa and Annie want to know how long conquering the galaxy will take. Annie suggests dividing the total number of days of work by the number of workers, which is two. What lower bound on the time to conquer the galaxy does this give, and why might the actual time required be greater?

(c) Lisa proposes a different method for determining the duration of their project. She suggests looking at the duration of the *critical path*, the most time-consuming sequence of tasks such that each depends on the one before. What lower bound does this give, and why might it also be too low?

(d) What is the minimum number of days that Lisa and Annie need to conquer the galaxy? No proof is required.

Problem 3.
Sauron finds that conquering Middle Earth breaks down into a bunch of tasks. Each task can be completed by a horrible creature called a *Ringwraith* in exactly one week. Sauron realizes the prerequisite structure among the tasks defines a DAG. He has n tasks in his DAG, with a maximum length chain of t tasks.

(a) Sauron is trying to describe various features of his scheduling problem using standard terminology. For each feature below, indicate the number of the corresponding term.

Standard Terminology

1.	Indirect prerequisite	2.	Topological sort
3.	Chain	4.	Antichain
5.	Size of the largest antichain	6.	Size of the smallest antichain
7.	Length of the longest chain	8.	Length of the shortest chain

1. A set of tasks that can be worked on simultaneously.
2. A possible order in which all the tasks could be completed, if only one Ringwraith were available.
3. The minimum number of weeks required to complete all tasks, if an unlimited number of Ringwraiths were available.

(b) If Sauron is lucky, he will be able to get away with a small crew of Ringwraiths. Write a simple formula involving n and t for the smallest number of Ringwraiths that could possibly be able to complete all n tasks in t weeks. (Do not make any additional assumptions about the relative sizes of n and t besides $t \leq n$.) Given any n and t, describe a DAG that can be completed in t weeks using this number of Ringwraiths.

(c) On the other hand, if Sauron is unlucky, he may need a large crew of Ringwraiths in order to conquer Middle Earth in t weeks. Write a simple formula involving n and t for the largest number of Ringwraiths that Sauron would ever need in order to be sure of completing all n tasks in t weeks—no matter how unlucky he was. Given any n and t, describe a DAG that can be completed in t weeks and requires this number of Ringwraiths.

Problem 4.
If a and b are distinct nodes of a digraph, then a is said to *cover* b if there is an edge from a to b and every path from a to b includes this edge. If a covers b, the edge from a to b is called a *covering edge*.

(a) What are the covering edges in the DAG in Figure 2?

(b) Let covering (D) be the subgraph of D consisting of only the covering edges. Suppose D is a finite DAG. Explain why covering (D) has the same positive walk relation as D.

Hint: Consider *longest* paths between a pair of vertices.

(c) Show that if two *DAG*'s have the same positive walk relation, then they have the same set of covering edges.

(d) Conclude that covering (D) is the *unique* DAG with the smallest number of edges among all digraphs with the same positive walk relation as D.

The following examples show that the above results don't work in general for digraphs with cycles.

(e) Describe two graphs with vertices $\{1, 2\}$ which have the same set of covering edges, but not the same positive walk relation (*Hint:* Self-loops.)

(f) (i) The *complete digraph* without self-loops on vertices $1, 2, 3$ has edges between every two distinct vertices. What are its covering edges?

(ii) What are the covering edges of the graph with vertices $1, 2, 3$ and edges $\langle 1 \to 2 \rangle$, $\langle 2 \to 3 \rangle$, $\langle 3 \to 1 \rangle$?

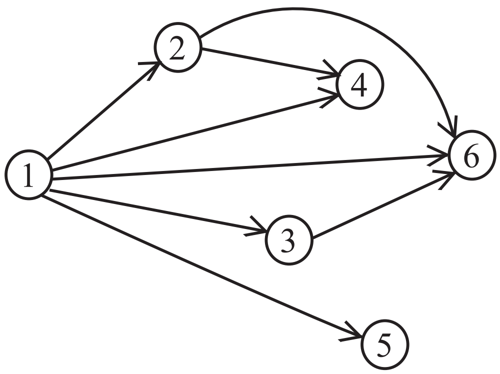

Figure 2 DAG with edges not needed in paths

(iii) What about their positive walk relations?