In-Class Problems Week 4, Mon.

Problem 1.
Prove by induction:

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}, \tag{1}$$

for all $n > 1$.

Problem 2. (a) Prove by induction that a $2^n \times 2^n$ courtyard with a 1×1 statue of Bill in *a corner* can be covered with L-shaped tiles. (Do not assume or reprove the (stronger) result of Theorem 5.1.2 that Bill can be placed anywhere. The point of this problem is to show a different induction hypothesis that works.)

(b) Use the result of part (a) to prove the original claim that there is a tiling with Bill in the middle.

Problem 3.
Any amount of 12 or more cents postage can be made using only 3¢ and 7¢ stamps. Prove this *by induction* using the induction hypothesis

$$S(n) ::= n + 12 \text{ cents postage can be made using only 3¢ and 7¢ stamps.}$$

Problem 4.
The following Lemma is true, but the *proof* given for it below is defective. Pinpoint *exactly* where the proof first makes an unjustified step and explain why it is unjustified.

Lemma. *For any prime p and positive integers $n, x_1, x_2, \ldots, x_n$, if $p \mid x_1 x_2 \ldots x_n$, then $p \mid x_i$ for some $1 \le i \le n$.*

Bogus proof. Proof by strong induction on n. The induction hypothesis, $P(n)$, is that Lemma holds for n.

Base case $n = 1$: When $n = 1$, we have $p \mid x_1$, therefore we can let $i = 1$ and conclude $p \mid x_i$.

Induction step: Now assuming the claim holds for all $k \le n$, we must prove it for $n + 1$.

So suppose $p \mid x_1 x_2 \cdots x_{n+1}$. Let $y_n = x_n x_{n+1}$, so $x_1 x_2 \cdots x_{n+1} = x_1 x_2 \cdots x_{n-1} y_n$. Since the righthand side of this equality is a product of n terms, we have by induction that p divides one of them. If $p \mid x_i$ for some $i < n$, then we have the desired i. Otherwise $p \mid y_n$. But since y_n is a product of the two terms x_n, x_{n+1}, we have by strong induction that p divides one of them. So in this case $p \mid x_i$ for $i = n$ or $i = n + 1$. ■