In-Class Problems Week 3, Wed.

Problem 1.
Set Formulas and Propositional Formulas.

(a) Verify that the propositional formula $(P \text{ AND } \overline{Q}) \text{ OR } (P \text{ AND } Q)$ is equivalent to P.

(b) Prove that[1]

$$A = (A - B) \cup (A \cap B)$$

for all sets, A, B, by showing

$$x \in A \text{ IFF } x \in (A - B) \cup (A \cap B)$$

for all elements, x, using the equivalence of part (a) in a chain of IFF's.

Problem 2.
A *formula of set theory* is a predicate formula that only uses the predicate "$x \in y$." The domain of discourse is the collection of sets, and "$x \in y$" is interpreted to mean the set x is one of the elements in the set y.

For example, since x and y are the same set iff they have the same members, here's how we can express equality of x and y with a formula of set theory:

$$(x = y) ::= \forall z. (z \in x \text{ IFF } z \in y). \tag{1}$$

Express each of the following assertions about sets by a formula of set theory.

(a) $x = \emptyset$.

(b) $x = \{y, z\}$.

(c) $x \subseteq y$. (x is a subset of y that might equal y.)

Now we can explain how to express "x is a proper subset of y" as a set theory formula using things we already know how to express. Namely, letting "$x \neq y$" abbreviate $\text{NOT}(x = y)$, the expression

$$(x \subseteq y \text{ AND } x \neq y),$$

describes a formula of set theory that means $x \subset y$.

From here on, feel free to use any previously expressed property in describing formulas for the following:

(d) $x = y \cup z$.

(e) $x = y - z$.

[1] The *set difference*, $A - B$, of sets A and B is

$$A - B ::= \{a \in A \mid a \notin B\}.$$

(f) $x = \text{pow}(y)$.

(g) $x = \bigcup_{z \in y} z$.

This means that y is supposed to be a collection of sets, and x is the union of all them. A more concise notation for "$\bigcup_{z \in y} z$' is simply "$\bigcup y$."

Problem 3.

Forming a pair (a, b) of items a and b is a mathematical operation that we can safely take for granted. But when we're trying to show how all of mathematics can be reduced to set theory, we need a way to represent the pair (a, b) as a set.

(a) Explain why representing (a, b) by $\{a, b\}$ won't work.

(b) Explain why representing (a, b) by $\{a, \{b\}\}$ won't work either. *Hint:* What pair does $\{\{1\}, \{2\}\}$ represent?

(c) Define

$$\text{pair}(a, b) ::= \{a, \{a, b\}\}.$$

Explain why representing (a, b) as $\text{pair}(a, b)$ uniquely determines a and b. *Hint:* Sets can't be indirect members of themselves: $a \in a$ never holds for any set a, and neither can $a \in b \in a$ hold for any b.

Problem 4.

Subset take-away[2] is a two player game played with a finite set, A, of numbers. Players alternately choose nonempty subsets of A with the conditions that a player may not choose

- the whole set A, or
- any set containing a set that was named earlier.

The first player who is unable to move loses the game.

For example, if the size of A is one, then there are no legal moves and the second player wins. If A has exactly two elements, then the only legal moves are the two one-element subsets of A. Each is a good reply to the other, and so once again the second player wins.

The first interesting case is when A has three elements. This time, if the first player picks a subset with one element, the second player picks the subset with the other two elements. If the first player picks a subset with two elements, the second player picks the subset whose sole member is the third element. In both cases, these moves lead to a situation that is the same as the start of a game on a set with two elements, and thus leads to a win for the second player.

Verify that when A has four elements, the second player still has a winning strategy.[3]

[2] From Christenson & Tilford, *David Gale's Subset Takeaway Game, American Mathematical Monthly, Oct. 1997*

[3] David Gale worked out some of the properties of this game and conjectured that the second player wins the game for any set A. This remains an open problem.