In-Class Problems Week 3, Tue.

Problem 1.
For each of the logical formulas, indicate whether or not it is true when the domain of discourse is $\mathbb{N}$, (the nonnegative integers 0, 1, 2, ...), $\mathbb{Z}$ (the integers), $\mathbb{Q}$ (the rationals), $\mathbb{R}$ (the real numbers), and $\mathbb{C}$ (the complex numbers). Add a brief explanation to the few cases that merit one.

$$\exists x.\, x^2 = 2$$
$$\forall x. \exists y.\, x^2 = y$$
$$\forall y. \exists x.\, x^2 = y$$
$$\forall x \neq 0. \exists y.\, xy = 1$$
$$\exists x. \exists y.\, x + 2y = 2 \text{ AND } 2x + 4y = 5$$

Problem 2.
The goal of this problem is to translate some assertions about binary strings into logic notation. The domain of discourse is the set of all finite-length binary strings: λ, 0, 1, 00, 01, 10, 11, 000, 001, ... (Here λ denotes the empty string.) In your translations, you may use all the ordinary logic symbols (including =), variables, and the binary symbols $\texttt{0}$, $\texttt{1}$ denoting 0, 1.

A string like $\texttt{01}x\texttt{0}y$ of binary symbols and variables denotes the *concatenation* of the symbols and the binary strings represented by the variables. For example, if the value of x is $\texttt{011}$ and the value of y is $\texttt{1111}$, then the value of $\texttt{01}x\texttt{0}y$ is the binary string $\texttt{0101101111}$.

Here are some examples of formulas and their English translations. Names for these predicates are listed in the third column so that you can reuse them in your solutions (as we do in the definition of the predicate NO-1S below).

Meaning	Formula	Name
x is a prefix of y	$\exists z\ (xz = y)$	PREFIX(x, y)
x is a substring of y	$\exists u \exists v\ (uxv = y)$	SUBSTRING(x, y)
x is empty or a string of 0’s	NOT(SUBSTRING$(\texttt{1}, x)$)	NO-1S(x)

(a) x consists of three copies of some string.

(b) x is an even-length string of 0’s.

(c) x does not contain both a 0 and a 1.

(d) x is the binary representation of $2^k + 1$ for some integer $k \geq 0$.

(e) An elegant, slightly trickier way to define NO-1S(x) is:

$$\text{PREFIX}(x, \texttt{0}x). \tag{*}$$

Explain why (*) is true only when x is a string of 0’s.

Problem 3.
Translate the following sentence into a predicate formula:

> There is a student who has e-mailed at most two other people in the class, besides possibly himself.

The domain of discourse should be the set of students in the class; in addition, the only predicates that you may use are

- equality, and
- $E(x, y)$, meaning that "x has sent e-mail to y."

Problem 4.
Provide a counter model for the implication that is not valid. Informally explain why the other one is valid.

1. $\forall x. \exists y. P(x, y)$ IMPLIES $\exists y. \forall x. P(x, y)$
2. $\exists y. \forall x. P(x, y)$ IMPLIES $\forall x. \exists y. P(x, y)$

Supplemental Problem[1]

Problem 5.
A certain cabal within the Math for Computer Science course staff is plotting to make the final exam *ridiculously hard*. ("Problem 1. Prove the Poincare Conjecture starting from the axioms of ZFC. Express your answer in khipu—the knot language of the Incas.") The only way to stop their evil plan is to determine exactly who is in the cabal. The course staff consists of seven people:

$$\{\text{Adam, Tom, Albert, Annie, Ben, Elizabeth, Siggi}\}$$

The cabal is a subset of these seven. A membership roster has been found and appears below, but it is deviously encrypted in logic notation. The predicate cabal indicates who is in the cabal; that is, cabal(x) is true if and only if x is a member. Translate each statement below into English and deduce who is in the cabal.

(a) $\exists x, y, z. (x \neq y \text{ AND } x \neq z \text{ AND } y \neq z \text{ AND } \text{cabal}(x) \text{ AND } \text{cabal}(y) \text{ AND } \text{cabal}(z))$

(b) NOT(cabal(Siggi) AND cabal(Annie))

(c) cabal(Elizabeth) IMPLIES $\forall x.$ cabal(x)

(d) cabal(Annie) IMPLIES cabal(Siggi)

(e) (cabal(Ben) OR cabal(Albert)) IMPLIES NOT(cabal(Tom))

(f) (cabal(Ben) OR cabal(Siggi)) IMPLIES NOT(cabal(Adam))

(g) Now use these facts to explain exactly who is on the cabal and why.

[1] There is no need to study supplemental problems when preparing for quizzes or exams.