In-Class Problems Week 3, Fri.

Problem 1.
The inverse, $R^{-1}$, of a binary relation, $R$, from $A$ to $B$, is the relation from $B$ to $A$ defined by:

$$b R^{-1} a \iff a R b.$$ 

In other words, you get the diagram for $R^{-1}$ from $R$ by “reversing the arrows” in the diagram describing $R$. Now many of the relational properties of $R$ correspond to different properties of $R^{-1}$. For example, $R$ is total iff $R^{-1}$ is a surjection.

Fill in the remaining entries in this table:

<table>
<thead>
<tr>
<th>$R$ is</th>
<th>$R^{-1}$ is</th>
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<tbody>
<tr>
<td>total</td>
<td>a surjection</td>
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<tr>
<td>a function</td>
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<tr>
<td>a surjection</td>
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<tr>
<td>an injection</td>
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<td>a bijection</td>
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*Hint*: Explain what’s going on in terms of “arrows” from $A$ to $B$ in the diagram for $R$.

Arrow Properties

**Definition.** A binary relation, $R$ is

- is a *function* when it has the $[\leq 1 \text{ arrow out}]$ property.
- is *surjective* when it has the $[\geq 1 \text{ arrows in}]$ property. That is, every point in the righthand, codomain column has at least one arrow pointing to it.
- is *total* when it has the $[\geq 1 \text{ arrows out}]$ property.
- is *injective* when it has the $[\leq 1 \text{ arrow in}]$ property.
- is *bijective* when it has both the $[= 1 \text{ arrow out}]$ and the $[= 1 \text{ arrow in}]$ property.

Problem 2.
Let $A = \{a_0, a_1, \ldots, a_{n-1}\}$ be a set of size $n$, and $B = \{b_0, b_1, \ldots, b_{m-1}\}$ a set of size $m$. Prove that $|A \times B| = mn$ by defining a simple bijection from $A \times B$ to the nonnegative integers from 0 to $mn - 1$.

Problem 3.
Assume $f : A \to B$ is total function, and $A$ is finite. Replace the $\ast$ with one of $\leq, =, \geq$ to produce the strongest correct version of the following statements:
(a) \(|f(A)| \star |B|\).

(b) If \(f\) is a surjection, then \(|A| \star |B|\).

(c) If \(f\) is a surjection, then \(|f(A)| \star |B|\).

(d) If \(f\) is an injection, then \(|f(A)| \star |A|\).

(e) If \(f\) is a bijection, then \(|A| \star |B|\).

**Problem 4.**
Let \(R : A \to B\) be a binary relation. Use an arrow counting argument to prove the following generalization of the Mapping Rule 1.

**Lemma.** If \(R\) is a function, and \(X \subseteq A\), then
\[|X| \geq |R(X)|.\]

**Problem 5.** (a) Prove that if \(A \text{ surj } B\) and \(B \text{ surj } C\), then \(A \text{ surj } C\).

(b) Explain why \(A \text{ surj } B\) iff \(B \text{ inj } A\).

(c) Conclude from (a) and (b) that if \(A \text{ inj } B\) and \(B \text{ inj } C\), then \(A \text{ inj } C\).

(d) Explain why \(A \text{ inj } B\) iff there is a total injective function \((\geq 1 \text{ out}, \leq 1 \text{ in})\) from \(A\) to \(B\). \(^1\)

\(^1\)The official definition of \(\text{inj}\) is with a total injective relation \((\geq 1 \text{ out}, \leq 1 \text{ in})\)