In-Class Problems Week 2, Wed.

Problem 1.
The proof below uses the Well Ordering Principle to prove that every amount of postage that can be assembled using only 6 cent and 15 cent stamps, is divisible by 3. Let the notation "$j \mid k$" indicate that integer j is a divisor of integer k, and let $S(n)$ mean that exactly n cents postage can be assembled using only 6 and 15 cent stamps. Then the proof shows that

$$S(n) \text{ IMPLIES } 3 \mid n, \quad \text{for all nonnegative integers } n. \tag{1}$$

Fill in the missing portions (indicated by "...") of the following proof of (1).

> Let C be the set of *counterexamples* to (1), namely[1]
>
> $$C ::= \{n \mid \ldots\}$$
>
> Assume for the purpose of obtaining a contradiction that C is nonempty. Then by the WOP, there is a smallest number, $m \in C$. This m must be positive because....
>
> But if $S(m)$ holds and m is positive, then $S(m-6)$ or $S(m-15)$ must hold, because.....
>
> So suppose $S(m-6)$ holds. Then $3 \mid (m-6)$, because....
>
> But if $3 \mid (m-6)$, then $3 \mid m$, because...,
>
> contradicting the fact that m is a counterexample.
>
> Next, if $S(m-15)$ holds, we arrive at a contradiction in the same way. Since we get a contradiction in both cases, we conclude that...
>
> which proves that (1) holds.

[1] The notation "$\{n \mid \ldots\}$" means "the set of elements, n, such that ...."

Problem 2.
Use the *Well Ordering Principle* [2] to prove that

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2}$$

for all nonnegative integers, n.

Problem 3.
Euler's Conjecture in 1769 was that there are no positive integer solutions to the equation

$$a^4 + b^4 + c^4 = d^4.$$

Integer values for a, b, c, d that do satisfy this equation were first discovered in 1986. So Euler guessed wrong, but it took more than two centuries to demonstrate his mistake.

Now let's consider Lehman's equation, similar to Euler's but with some coefficients:

$$8a^4 + 4b^4 + 2c^4 = d^4 \tag{3}$$

Prove that Lehman's equation (3) really does not have any positive integer solutions.

Hint: Consider the minimum value of a among all possible solutions to (3).

Problem 4.
You are given a series of envelopes, respectively containing $1, 2, 4, \ldots, 2^m$ dollars. Define

> **Property** m: For any nonnegative integer less than 2^{m+1}, there is a selection of envelopes whose contents add up to *exactly* that number of dollars.

Use the Well Ordering Principle (WOP) to prove that Property m holds for all nonnegative integers m.

Hint: Consider two cases: first, when the target number of dollars is less than 2^m and second, when the target is at least 2^m.

Problem 5.
Use the Well Ordering Principle to prove that any integer greater than or equal to 30 can be represented as the sum of nonnegative integer multiples of 6, 10, and 15.

Hint: Use the template for WOP proofs to ensure partial credit. Verify that integers in the interval [30..35] are sums of nonnegative integer multiples of 6, 10, and 15.

[2] Proofs by other methods such as induction or by appeal to known formulas for similar sums will not receive credit.