

In-Class Problems Week 1, Wed.

Problem 1.

The Pythagorean Theorem says that if a and b are the lengths of the sides of a right triangle, and c is the length of its hypotenuse, then

$$a^2 + b^2 = c^2.$$

This theorem is so fundamental and familiar that we generally take it for granted. But just being familiar doesn't justify calling it "obvious"—witness the fact that people have felt the need to devise different proofs of it for millenia.¹ In this problem we'll examine a particularly simple "proof without words" of the theorem.

Here's the strategy. Suppose you are given four different colored copies of a right triangle with sides of lengths a , b , and c , along with a suitably sized square, as shown in Figure 1.

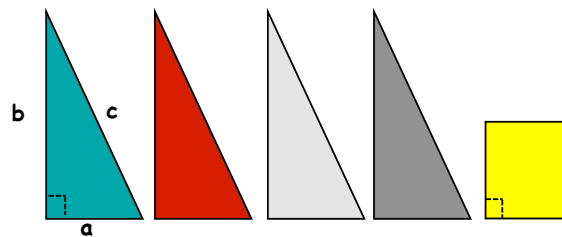


Figure 1 Right triangles and square.

(a) You will first arrange the square and four triangles so they form a $c \times c$ square. From this arrangement you will see that the square is $(b - a) \times (b - a)$.

(b) You will then arrange the same shapes so they form two squares, one $a \times a$ and the other $b \times b$.


You know that the area of an $s \times s$ square is s^2 . So appealing to the principle that

Area is Preserved by Rearranging,

you can now conclude that $a^2 + b^2 = c^2$, as claimed.

This really is an elegant and convincing proof of the Pythagorean Theorem, but it has some worrisome features. One concern is that there might be something special about the shape of these particular triangles and square that makes the rearranging possible—for example, suppose $a = b$

(c) How would you respond to this concern?

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¹Over a hundred different proofs are listed on the mathematics website <http://www.cut-the-knot.org/pythagoras/>.

(d) Another concern is that a number of facts about right triangles, squares and lines are being *implicitly* assumed in justifying the rearrangements into squares. Enumerate some of these assumed facts.

Problem 2.

What's going on here?!

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = (\sqrt{-1})^2 = -1.$$

(a) Precisely identify and explain the mistake(s) in this *bogus* proof.

(b) Prove (correctly) that if $1 = -1$, then $2 = 1$.

(c) Every *positive* real number, r , has two square roots, one positive and the other negative. The standard convention is that the expression \sqrt{r} refers to the *positive* square root of r . Assuming familiar properties of multiplication of real numbers, prove that for positive real numbers r and s ,

$$\sqrt{rs} = \sqrt{r}\sqrt{s}.$$

Problem 3.

Identify exactly where the bugs are in each of the following bogus proofs.²

(a) **Bogus Claim:** $1/8 > 1/4$.

Bogus proof.

$$\begin{aligned} 3 &> 2 \\ 3 \log_{10}(1/2) &> 2 \log_{10}(1/2) \\ \log_{10}(1/2)^3 &> \log_{10}(1/2)^2 \\ (1/2)^3 &> (1/2)^2, \end{aligned}$$

and the claim now follows by the rules for multiplying fractions. ■

(b) *Bogus proof:* $1\text{¢} = \$0.01 = (\$0.1)^2 = (10\text{¢})^2 = 100\text{¢} = \1 . ■

(c) **Bogus Claim:** If a and b are two equal real numbers, then $a = 0$.

Bogus proof.

$$\begin{aligned} a &= b \\ a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a - b)(a + b) &= (a - b)b \\ a + b &= b \\ a &= 0. \end{aligned}$$

■

²From [42], *Twenty Years Before the Blackboard* by Michael Stueben and Diane Sandford

Problem 4.

It's a fact that the Arithmetic Mean is at least as large as the Geometric Mean, namely,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

for all nonnegative real numbers a and b . But there's something objectionable about the following proof of this fact. What's the objection, and how would you fix it?

Bogus proof.

$$\begin{aligned} \frac{a+b}{2} &\stackrel{?}{\geq} \sqrt{ab}, && \text{so} \\ a+b &\stackrel{?}{\geq} 2\sqrt{ab}, && \text{so} \\ a^2 + 2ab + b^2 &\stackrel{?}{\geq} 4ab, && \text{so} \\ a^2 - 2ab + b^2 &\stackrel{?}{\geq} 0, && \text{so} \\ (a-b)^2 &\geq 0 && \text{which we know is true.} \end{aligned}$$

The last statement is true because $a-b$ is a real number, and the square of a real number is never negative. This proves the claim. ■

Optional (and controversial)**Problem 5.**

Albert announces to his class that he plans to surprise them with a quiz sometime next week.

His students first wonder if the quiz could be on Friday of next week. They reason that it can't: if Albert didn't give the quiz *before* Friday, then by midnight Thursday, they would know the quiz had to be on Friday, and so the quiz wouldn't be a surprise any more.

Next the students wonder whether Albert could give the surprise quiz Thursday. They observe that if the quiz wasn't given *before* Thursday, it would have to be given *on* the Thursday, since they already know it can't be given on Friday. But having figured that out, it wouldn't be a surprise if the quiz was on Thursday either. Similarly, the students reason that the quiz can't be on Wednesday, Tuesday, or Monday. Namely, it's impossible for Albert to give a surprise quiz next week. All the students now relax, having concluded that Albert must have been bluffing. And since no one expects the quiz, that's why, when Albert gives it on Tuesday next week, it really is a surprise!

What, if anything, do you think is wrong with the students' reasoning?