

In-Class Problems Week 1, Fri.

Problem 1.

Prove that if $a \cdot b = n$, then either a or b must be $\leq \sqrt{n}$, where a, b , and n are nonnegative real numbers.
Hint: by contradiction, Section 1.8.

Problem 2.

Generalize the proof of Theorem 1.8.1 repeated below that $\sqrt{2}$ is irrational¹. For example, how about $\sqrt{3}$?

Theorem. $\sqrt{2}$ is an irrational number.

Proof. The proof is by contradiction: assume that $\sqrt{2}$ is rational, that is,

$$\sqrt{2} = \frac{n}{d}, \tag{1}$$

where n and d are integers. Now consider the smallest such positive integer denominator, d . We will prove in a moment that the numerator, n , and the denominator, d , are both even. This implies that

$$\frac{n/2}{d/2}$$

is a fraction equal to $\sqrt{2}$ with a smaller positive integer denominator, a contradiction.

Since the assumption that $\sqrt{2}$ is rational leads to this contradiction, the assumption must be false. That is, $\sqrt{2}$ is indeed irrational. This italicized comment on the implication of the contradiction normally goes without saying, but since this is an early example of proof by contradiction, we've said it.

To prove that n and d have 2 as a common factor, we start by squaring both sides of (1) and get $2 = n^2/d^2$, so

$$2d^2 = n^2. \tag{2}$$

So 2 is a factor of n^2 , which is only possible if 2 is in fact a factor of n .

This means that $n = 2k$ for some integer, k , so

$$n^2 = (2k)^2 = 4k^2. \tag{3}$$

Combining (2) and (3) gives $2d^2 = 4k^2$, so

$$d^2 = 2k^2. \tag{4}$$

So 2 is a factor of d^2 , which again is only possible if 2 is in fact also a factor of d , as claimed. ■

¹Remember that an irrational number is a number that cannot be expressed as a ratio of two integers.

Problem 3.

If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering $\sqrt{2}^{\sqrt{2}}$ and arguing by cases.

Problem 4.

The fact that there are irrational numbers a, b such that a^b is rational was proved earlier by cases. Unfortunately, that proof was *nonconstructive*: it didn't reveal a specific pair, a, b , with this property. But in fact, it's easy to do this: let $a ::= \sqrt{2}$ and $b ::= 2 \log_2 3$.

We know $a = \sqrt{2}$ is irrational, and $a^b = 3$ by definition. Finish the proof that these values for a, b work, by showing that $2 \log_2 3$ is irrational.