Massachusetts Institute of Technology
6.042J/18.062J, Spring '15: Mathematics for Computer Science — May 6
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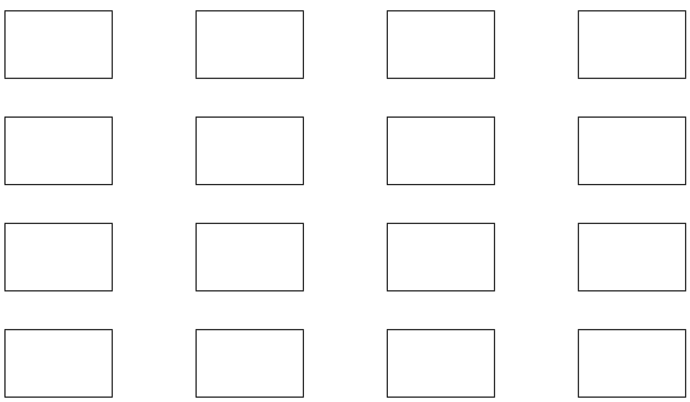
In-Class Problems Week 13, Wed.

Problem 1.
Here's a dice game with maximum payoff k: make three independent rolls of a fair die, and if you roll a six

- no times, then you lose 1 dollar;
- exactly once, then you win 1 dollar;
- exactly twice, then you win 2 dollars;
- all three times, then you win k dollars.

For what value of k is this game fair?[1]

Problem 2.
A classroom has sixteen desks in a 4×4 arrangement as shown below.



If there is a girl in front, behind, to the left, or to the right of a boy, then the two *flirt*. One student may be in multiple flirting couples; for example, a student in a corner of the classroom can flirt with up to two others, while a student in the center can flirt with as many as four others. Suppose that desks are occupied mutually independently by boys and girls with equal probability. What is the expected number of flirting couples? *Hint:* Linearity.

Problem 3. **(a)** Suppose we flip a fair coin and let $N_{\mathtt{TT}}$ be the number of flips until the first time two consecutive Tails appear. What is $\mathrm{Ex}[N_{\mathtt{TT}}]$?

[1] This game is actually offered in casinos with $k = 3$, where it is called Carnival Dice.

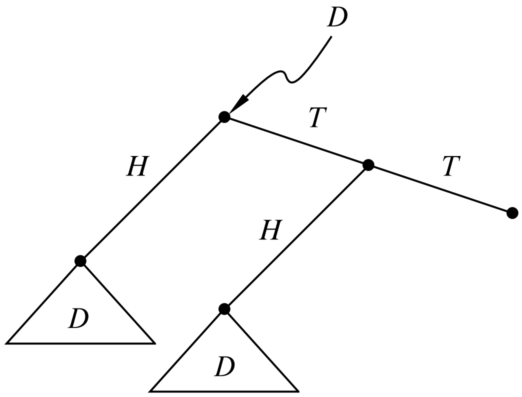


Figure 1 Sample space tree for coin toss until two consecutive tails.

Hint: Let D be the tree diagram for this process. Explain why D can be described by the tree in Figure 1. Use the **Law of Total Expectation**: Let R be a random variable and $A_1, A_2, \ldots,$ be a partition of the sample space. Then

$$\mathrm{Ex}[R] = \sum_i \mathrm{Ex}[R \mid A_i] \Pr[A_i].$$

(b) Let $N_{\texttt{TH}}$ be the number of flips until a Tail immediately followed by a Head comes up. What is $\mathrm{Ex}[N_{\texttt{TH}}]$?

(c) Suppose we now play a game: flip a fair coin until either `TT` or `TH` occurs. You win if `TT` comes up first, and lose if `TH` comes up first. Since `TT` takes 50% longer on average to turn up, your opponent agrees that he has the advantage. So you tell him you're willing to play if you pay him \$5 when he wins, and he pays you with a mere 20% premium—that is \$6—when you win.

If you do this, you're sneakily taking advantage of your opponent's untrained intuition, since you've gotten him to agree to unfair odds. What is your expected profit per game?

Problem 4.
Let T be a positive integer valued random variable such that

$$\mathrm{PDF}_T(n) = \frac{1}{an^2},$$

where

$$a ::= \sum_{n \in \mathbb{Z}^+} \frac{1}{n^2}.$$

(a) Prove that $\mathrm{Ex}[T]$ is infinite.

(b) Prove that $\mathrm{Ex}[\sqrt{T}]$ is finite.

Supplementary Problems

Problem 5.
Suppose there are 4 desks in a classroom, laid out in the corners of a square with corners 1 2 3 and 4.

Each desk is occupied by a male with probability $p > 0$ or a female with probability $q ::= 1 - p > 0$. A male and a female *flirt* when they occupy desks in adjacent corners of the square. Let $I_{12}, I_{23}, I_{34}, I_{41}$ be the indicator variables that there is a flirting couple at the indicated adjacent desks.

(a) Show that if $p = q$ then the events $I_{12} = 1$ and $I_{23} = 1$ are independent.

(b) Show rigorously that if the events $I_{12} = 1$ and $I_{23} = 1$ are independent then $p = q$. Hint: work from the definition of independence, set up an equation and solve.

(c) What is the expected number of flirting couples in terms of p and q?

Problem 6.
Justify each line of the following proof that if R_1 and R_2 are *independent*, then

$$\text{Ex}[R_1 \cdot R_2] = \text{Ex}[R_1] \cdot \text{Ex}[R_2].$$

Proof.

$$\begin{aligned}
&\text{Ex}[R_1 \cdot R_2] \\
&= \sum_{r \in \text{range}(R_1 \cdot R_2)} r \cdot \Pr[R_1 \cdot R_2 = r] \\
&= \sum_{r_i \in \text{range}(R_i)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \\
&= \sum_{r_1 \in \text{range}(R_1)} \sum_{r_2 \in \text{range}(R_2)} r_1 r_2 \cdot \Pr[R_1 = r_1] \cdot \Pr[R_2 = r_2] \\
&= \sum_{r_1 \in \text{range}(R_1)} \left(r_1 \Pr[R_1 = r_1] \cdot \sum_{r_2 \in \text{range}(R_2)} r_2 \Pr[R_2 = r_2] \right) \\
&= \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \cdot \text{Ex}[R_2] \\
&= \text{Ex}[R_2] \cdot \sum_{r_1 \in \text{range}(R_1)} r_1 \Pr[R_1 = r_1] \\
&= \text{Ex}[R_2] \cdot \text{Ex}[R_1].
\end{aligned}$$

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