In-Class Problems Week 12, Fri.

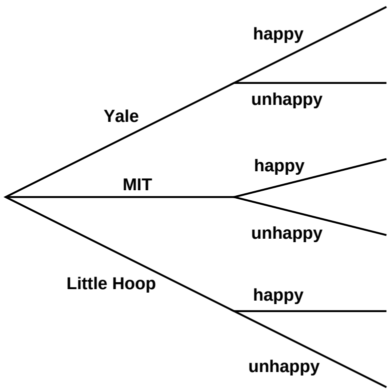
Problem 1.
Sally Smart just graduated from high school. She was accepted to three reputable colleges.

- With probability $4/12$, she attends Yale.
- With probability $5/12$, she attends MIT.
- With probability $3/12$, she attends Little Hoop Community College.

Sally is either happy or unhappy in college.

- If she attends Yale, she is happy with probability $4/12$.
- If she attends MIT, she is happy with probability $7/12$.
- If she attends Little Hoop, she is happy with probability $11/12$.

(a) A tree diagram to help Sally project her chance at happiness is shown below. On the diagram, fill in the edge probabilities, and at each leaf write the probability of the corresponding outcome.



(b) What is the probability that Sally is happy in college?

(c) What is the probability that Sally attends Yale, given that she is happy in college?

(d) Show that the event that Sally attends Yale **is not** independent of the event that she is happy.

(e) Show that the event that Sally attends MIT **is** independent of the event that she is happy.

Problem 2.
Suppose you flip three fair, mutually independent coins. Define the following events:

- Let A be the event that *the first* coin is heads.
- Let B be the event that *the second* coin is heads.
- Let C be the event that *the third* coin is heads.
- Let D be the event that *an even number of* coins are heads.

(a) Use the four step method to determine the probability space for this experiment and the probability of each of A, B, C, D.

(b) Show that these events are not mutually independent.

(c) Show that they are 3-way independent.

Problem 3.
Graphs, Logic & Probability

Let G be an undirected simple graph with $n > 3$ vertices. Let $E(x, y)$ mean that G has an edge between vertices x and y, and let $P(x, y)$ mean that there is a length 2 walk in G between x and y.

(a) Write a predicate-logic formula defining $P(x, y)$ in terms of $E(x, y)$.

For the following parts (b)–(d), let V be a fixed set of $n > 3$ vertices, and let G be a graph with these vertices constructed randomly as follows: for all distinct vertices $x, y \in V$, independently include edge $\langle x\text{—}y \rangle$ as an edge of G with probability p. In particular, $\Pr[E(x, y)] = p$ for all $x \neq y$.

(b) For distinct vertices w, x, y and z in V, circle the event pairs that are independent.

1. $E(w, x)$ versus $E(x, y)$
2. $[E(w, x)$ AND $E(w, y)]$ versus $[E(z, x)$ AND $E(z, y)]$
3. $E(x, y)$ versus $P(x, y)$
4. $P(w, x)$ versus $P(x, y)$
5. $P(w, x)$ versus $P(y, z)$

(c) Write a simple formula in terms of n and p for $\Pr[\text{NOT } P(x, y)]$, for distinct vertices x and y in V.

Hint: Use part (b), item 2.

(d) What is the probability that two distinct vertices x and y lie on a three-cycle in G? Answer with a simple expression in terms of p and r, where $r ::= \Pr[\text{NOT}(P(x, y))]$ is the correct answer to part (c).

Hint: Express x and y being on a three-cycle as a simple formula involving $E(x, y)$ and $P(x, y)$.

Supplemental Problem

Problem 4.
Let A, B, C be events. For each of the following statements, prove it or give a counterexample.

(a) If A is independent of B, then A is also independent of $\overline{B}$.

(b) If A is independent of B, and A is independent of C, then A is independent of $B \cap C$.

Hint: Choose A, B, C pairwise but not 3-way independent.

(c) If A is independent of B, and A is independent of C, then A is independent of $B \cup C$.

Hint: Part (b).

(d) If A is independent of B, and A is independent of C, and A is independent of $B \cap C$, then A is independent of $B \cup C$.